Table 20: Wear Data, Tire Experiment

<table>
<thead>
<tr>
<th>Tire</th>
<th>Compound</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>A</td>
</tr>
<tr>
<td>1</td>
<td>238</td>
</tr>
<tr>
<td>2</td>
<td>196</td>
</tr>
<tr>
<td>3</td>
<td>254</td>
</tr>
<tr>
<td>4</td>
<td>312</td>
</tr>
</tbody>
</table>

\(E(G_{1j} - G_{2j}) = \frac{1}{6}((3\gamma + a_{1} + a_{2} + a_{3} + 3\tau)\tau_{1} - \tau_{2} + 3\tau_{3} + 3\tau_{4})\)

\(\frac{1}{4}(127 + 3a_{1} + 3a_{2} + 3a_{3} + 3a_{4} + 3\tau_{1} + 3\tau_{2} + 3\tau_{3} + 3\tau_{4})\)

Let \(S_{j} = \frac{b_1}{b}G_{1j} - \frac{b_2}{b}G_{2j}\) and \(Q_{j} = S_{j} - \sum_{i} b_{i}n_{i}G_{ij}\).

\(E(Q_1) = 6E(S_{1} - G_{1j} - G_{2j}) = \frac{2}{3}j\sum_{i} b_{i}n_{i}\)

\(= (3\gamma + 3a_{1} + a_{2} + a_{3}) - \frac{1}{3}j(3\gamma + 3a_{1} + a_{2} + a_{3})\)

\(\frac{1}{2}(3\gamma + a_{1} + a_{2} + a_{3} + 3\tau_{1} + 3\tau_{2} + 3\tau_{3} + 3\tau_{4})\)

\(\Rightarrow E(\frac{Q_{j}}{\sqrt{j}}) = \tau_{j} - \tau \Rightarrow \tau_{j} = \left(\frac{Q_{j}}{\sqrt{j}}\right)\frac{\sqrt{n_{j}}}{\sqrt{\sum_{i} b_{i}n_{i}}}\)

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**Balanced Incomplete Block Design (BIBD)**

- A BIBD has \(t\) treatments, and \(b\) blocks of size \(k\).
- \(t > k\), each treatment replicated \(r\) times, such that each pair of treatments appear in the same number (denoted by \(\lambda\)) of blocks. (In the wear experiment, \(t = 4, b = 4, k = 3, r = 3\) and \(\lambda = 2\).)

Two basic relations:

(i) Count of \(EUs\): \(4 \times \frac{3}{b \times k} = \frac{3 \times 4}{r \times t}\)

(ii) Count of WBCs in \(A\) vs. other factors: \(3 \times \frac{2}{r \times (k - 1)} = \lambda \times (r - 1)\)

For given \(k, t, b\), a BIBD may or may not exist. When it does not, either adjust the values of \(k, t, b\) to get a BIBD, or if not possible, find a partially balanced incomplete block design (PBIBD) (which is not covered in the book). Tables of BIBD or PBIBD in books like Cochran and Cox (1957).

**Reading:** textbook, 3.8
Analysis of Covariance: Starch Experiment

- Data in Table 3.34 of textbook (p.129).
- Goal: To compare the three treatments (canna, corn, potato) for making starch film, \( y = \) break strength of film, covariate \( x = \) film thickness.
- Known that \( x \) affects \( y \) (thicker films are stronger); thickness cannot be controlled but are measured after films are made.
- Question: How to perform treatment comparisons by incorporating the effect of \( x \)?
- Model: 
  \[
  y_{ij} = \eta + \gamma \times x_{ij} + \tau_i + \varepsilon_{ij},
  \]
  where \( i = 1, \ldots, k \), \( j = 1, \ldots, n_i \), and
  \( \tau_i = \) \( i \)th treatment effect
  \( x_{ij} = \) covariate value.
  \( \gamma = \) regression coefficient for the \( x_{ij} \)
  \( \varepsilon_{ij} = \) independent \( \sim \text{N}(0, \sigma^2) \).
- Special cases of the model:
  1. When \( \gamma = 0 \) (i.e., \( x_{ij} \) not available or no \( x \) effect), one-way layout.
  2. When \( \tau_i = 0 \) (no treatment effect), simple linear regression.

Regression Model Approach

- Model:
  \[
  E(y_{ij}, x) = \eta + \gamma x_{ij} + \tau_i + \varepsilon_{ij}, \quad j = 1, \ldots, 13, \quad i = 1, 2, 3
  \]
  where
  \( \tau_1 = 0 \) (treatment codings)
  \( \eta = \) intercept,
  \( \gamma = \) regression coefficient for thickness,
  \( \tau_2 = \) canna vs. corn, and
  \( \tau_3 = \) canna vs. potato.
  (Write the model matrix for (6)).
- Run regression analysis in the usual way.
Regression Analysis of Starch Experiment

**Q:** how to perform sequential ANOVA for this case?

**Table 21: Tests, Starch Experiment**

<table>
<thead>
<tr>
<th>Effect</th>
<th>Estimate</th>
<th>Standard Error</th>
<th>t</th>
<th>p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>intercept</td>
<td>158.261</td>
<td>179.775</td>
<td>0.88</td>
<td>0.38</td>
</tr>
<tr>
<td>thickness</td>
<td>62.501</td>
<td>17.060</td>
<td>3.66</td>
<td>0.00</td>
</tr>
<tr>
<td>$\tau_2$</td>
<td>-83.666</td>
<td>86.095</td>
<td>0.97</td>
<td>0.34</td>
</tr>
<tr>
<td>$\tau_2$ vs corn</td>
<td>70.360</td>
<td>67.782</td>
<td>1.04</td>
<td>0.30</td>
</tr>
<tr>
<td>corn vs potato</td>
<td>154.026</td>
<td>107.762</td>
<td>1.43</td>
<td>0.16</td>
</tr>
</tbody>
</table>

Ho: $\tau_2 = \tau_3 \Leftrightarrow$ merge $\tau_2, \tau_3$ columns in $X$ (MM)

- In the table, corn vs potato = $\hat{\tau}_3 - \hat{\tau}_2 = 70.360 - (-83.666) = 154.026$.
- No pair of film types has any significant difference after adjusting for thickness effect. (So, how should the choice be made between the three film types?) Most of the variation is explained by the covariate thickness.

**Q:** What if we fit the model: $y$?

$y \sim \beta_0 + \text{starch} + \epsilon$

$Y = X_1 \beta_1 + X_2 \beta_2 + \epsilon$

**Multiple Comparisons**

1. obtain $t_{ij}^2 = \frac{\hat{u}_{ij} - \hat{u}_{ij}}{s.e.(\hat{u}_{ij} - \hat{u}_{ij})}$
2. determine critical value (or p-value)

- $Var(\hat{\tau}_3)$ and $Var(\hat{\tau}_2)$ can be obtained from regression output Table 21 (LNp.4-42).
- From (1.33) of textbook (p.22),

$$Var(\hat{\beta}) = \sigma^2 (X^T X)^{-1}.$$  

- Using this, for $H_0: \tau_2 = \tau_3$ (i.e., $\tau_3 - \tau_2 = 0$),

the variance of $\hat{\tau}_3 - \hat{\tau}_2$ can be found as

$$Var(\hat{\tau}_3 - \hat{\tau}_2) = Var(\hat{\tau}_3) + Var(\hat{\tau}_2) - 2 Cov(\hat{\tau}_3, \hat{\tau}_2).$$

- The degrees of freedom for the t-statistic is same as that of the residuals. The $p$-values for the three tests are given in Table 21 (LNp.4-42).
- For simultaneous testing, use adjusted $p$-values (LNp.4-24).
**ANOVA Table**

- **ANCOVA Table** → Apply sequential ANOVA to the model with covariate treated as block factor and usually not orthogonal to treatment factor.

\[
\begin{aligned}
&\text{ANOVA (}\, Y \sim B_0 + \text{thickness}_1 + \text{starch}) \\
&B_0 \quad (M1) \; Y \sim B_0 \\
&B_0 + \text{thickness}_1 \quad (M2) \; Y \sim B_0 + \text{thickness}_1 \\
&B_0 + \text{thickness}_1 + \text{starch} \quad (M3) \; Y \sim B_0 + \text{thickness}_1 + \text{starch}
\end{aligned}
\]

Note: covariate must appear before treatment factor.

Table 22: ANCOVA Table, Starch Experiment

<table>
<thead>
<tr>
<th>Source</th>
<th>Degrees of Freedom</th>
<th>Sum of Squares</th>
<th>Mean Squares</th>
<th>F</th>
</tr>
</thead>
<tbody>
<tr>
<td>thickness</td>
<td>1</td>
<td>2553357</td>
<td>2553357</td>
<td>94.19</td>
</tr>
<tr>
<td>starch</td>
<td>2</td>
<td>56725</td>
<td>28362</td>
<td>1.05</td>
</tr>
<tr>
<td>residual</td>
<td>45</td>
<td>1219940</td>
<td>27110</td>
<td></td>
</tr>
</tbody>
</table>

*(exercise)* Q. What if we use

\[
\text{ANOVA (}\, Y \sim B_0 + \text{starch} + \text{thickness}_2) \]

Note: It is possible that we are not interested in "starch" (as block), but are interested in the coefficient of "thickness" (as treatment).

Reading: textbook, 3.10

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**Example of Split-plot Design: Wood Experiment**

- **Experiment objective:** to study the water resistant property of wood.

- **Two treatment factors:**
  - **A** — wood pretreatments: \(a_1, a_2\);
    - qualitative, 2 levels
  - **B** — types of stain: \(b_1, b_2, b_3, b_4\).
    - qualitative, 4 levels

Completely randomized design: randomly apply the 8 level combinations of **A** and **B** to 8 wood panels, such as in Table 23.

- Problem: inconvenient to apply the pretreatments to a small wood panel.

Q: **How many times pretreatments \(a_1\) or \(a_2\) are applied?**

\[\text{Ans. 8 times} \rightarrow 8 \text{ EUs} \]

Reading: textbook, 3.10

Table 23: Completely Randomized Version of the Wood Experiment

<table>
<thead>
<tr>
<th>Run (EUs)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pretreatment (A)</td>
<td>(a_1)</td>
<td>(a_2)</td>
<td>(a_2)</td>
<td>(a_1)</td>
<td>(a_2)</td>
<td>(a_1)</td>
<td>(a_2)</td>
<td>(a_2)</td>
</tr>
<tr>
<td>Stain (B)</td>
<td>(b_2)</td>
<td>(b_4)</td>
<td>(b_1)</td>
<td>(b_1)</td>
<td>(b_3)</td>
<td>(b_4)</td>
<td>(b_3)</td>
<td>(b_2)</td>
</tr>
</tbody>
</table>

\[
\text{Q: What does } \varepsilon \text{ represent?}
\]

**COV (\(\varepsilon\)) = \sigma^2 I**

# of distinct level combinations of **A** and **B**

Jointly made by Jeff Wu (GT, USA) and S.-W. Cheng (NTHU, Taiwan)
### Split-plot Design

- Alternative Design: split-plot design in Table 24.

#### Table 24: Split-Plot Version of the Wood Experiment

<table>
<thead>
<tr>
<th></th>
<th>First panel</th>
<th>Second panel</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pretreated with</td>
<td>$a_1$</td>
<td>$a_2$</td>
</tr>
<tr>
<td></td>
<td>$b_3$, $b_5$, $b_4$, $b_1$</td>
<td>$b_7$, $b_1$, $b_4$, $b_3$</td>
</tr>
</tbody>
</table>

**Q:** How many times pretreatments ($a_1$ or $a_2$) are applied?

**Ans:** 2 times

- Justification: Easier to apply pretreatment to large wood panels.

*Exp’l units (Q: what are the exp’l units):*

- EUs for A and B are different
- EUs for A: large panels
- EUs for B: small panels
- Large panels are split into small panels

"Small panel" is nested in "large panel"

### Split-plot Design (Cont’d)

- Why is it called "plot"?

- Split-plot design (and the name) has its origin in agriculture.

- Some factors need to be applied to large plots, called whole plots. In the example, the two big wood panels to which pretreatment $a_1$ and $a_2$ are applied are whole plots.

- Split each whole plot into smaller plots, called subplots. In the example, the four small wood panels within the large panels are subplots.

- Wood Experiment:
  - $A$: whole-plot factor
  - $B$: subplot factor
  - 3 replications (treated as 3 blocks with random effects)
  - 6 whole plots (two large panels for $a_1$ and $a_2$ per replication)
  - 24 subplots (four small panels for $b_1$, $b_2$, $b_3$, and $b_4$ per large panel)

- Similar to the 1-way random effects model discussed in LNP 3-31~38

**Conceptual model:**

$y \sim B_0 + A + B + A \times B + \varepsilon^A + \varepsilon^E$

$\text{cov}(\varepsilon) + \varepsilon^S$
Data from the Wood Experiment

<table>
<thead>
<tr>
<th>Whole plot</th>
<th>Sub-plot</th>
<th>Pretreatment (A)</th>
<th>Stain type (B)</th>
<th>Replication (Rep)</th>
<th>Resistance (Y)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>(a_2)</td>
<td>(b_2)</td>
<td>(0)</td>
<td>53.5</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(a_2)</td>
<td>(b_1)</td>
<td>(0)</td>
<td>40.8</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(a_2)</td>
<td>(b_1)</td>
<td>(0)</td>
<td>43.0</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(a_2)</td>
<td>(b_3)</td>
<td>(0)</td>
<td>51.8</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(a_1)</td>
<td>(b_4)</td>
<td>(1)</td>
<td>45.3</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(a_1)</td>
<td>(b_2)</td>
<td>(1)</td>
<td>44.6</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(a_1)</td>
<td>(b_2)</td>
<td>(1)</td>
<td>45.9</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(a_1)</td>
<td>(b_4)</td>
<td>(2)</td>
<td>48.3</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(a_1)</td>
<td>(b_4)</td>
<td>(2)</td>
<td>60.9</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(a_1)</td>
<td>(b_2)</td>
<td>(2)</td>
<td>51.1</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(a_1)</td>
<td>(b_2)</td>
<td>(2)</td>
<td>51.4</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(a_1)</td>
<td>(b_1)</td>
<td>(3)</td>
<td>30.1</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(a_1)</td>
<td>(b_3)</td>
<td>(3)</td>
<td>34.4</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(a_1)</td>
<td>(b_3)</td>
<td>(3)</td>
<td>32.2</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(a_1)</td>
<td>(b_1)</td>
<td>(3)</td>
<td>52.8</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(a_1)</td>
<td>(b_1)</td>
<td>(3)</td>
<td>51.7</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(a_1)</td>
<td>(b_3)</td>
<td>(3)</td>
<td>55.3</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(a_1)</td>
<td>(b_3)</td>
<td>(3)</td>
<td>59.2</td>
</tr>
</tbody>
</table>

\(Y \sim N(\mu, \sigma^2)\)

Assume all \(\varepsilon\)s are independent.