The estimates of parameters in this model correspond only to the marginal totals \(y_{1+}, y_{+1},\) and \(y_{++}\).

The coding we use will determine exactly how the parameters relate to the margin totals, e.g., let \(\beta\) be an main effect of \(X_1\) that codes \(i_1\) and \(i_2\) categories as 0 (reference) and 1.

Insignificant factor, say \(X_1 \Rightarrow \pi_{1+} = \pi_{2+} = \cdots = \pi_{I+}\).

Joint independence (\(\{X_1, X_2\}\) and \(X_3\) are independent)

\[
\pi_{ijk} = \pi_{ij} \times \pi_{ik} \times \pi_{jk} \iff \pi_{ijk} = \pi_{ij} \times \pi_{jk} \times \pi_{ik}
\]

No need to be independent.

Conditional independence (\(X_1, X_2\) are independent given \(X_3\))

\[
\log(\pi_{ijk}) = \log(\pi_{i+j+k}) \iff \log(\pi_{i+j+k}) = \log(\pi_{i+k} + \pi_{j+k})
\]

Common factor in 2fi's

\[
Y \sim X_1 + X_2 + X_1 X_2 + X_3 = S_2 (\supset S_1)
\]

Note that \(X_3\) is fixed, but \(X_2\) is jointly independent of \(\{X_1, X_3\}\) implies that \(X_1, X_2\) are independent given \(X_3\).

Q: can this conditional independence imply independence between \(X_1\) and \(X_2\), i.e., \(\pi_{ij} = \pi_{i+} \pi_{j+}\)? (Ans: No. Check singular value decomposition in LNp.5-15)
Uniform association (UA) \[ \text{conditional independence} \Rightarrow \text{uniform association} \]

- Consider a model with all two-factor interactions
  \[ Y \sim X_1 + X_2 + X_3 + X_1 \cdot X_2 + X_1 \cdot X_3 + X_2 \cdot X_3 \equiv S_4 \\left( \supseteq S_3 \right) \]
- \( S_4 \) has no simple interpretation in terms of independence
  \( S_4 \) asserts that for every level of one variable, say \( X_3 \), we have the same association between \( X_1 \) and \( X_2 \).

\[ Y \sim X_1 + X_2 + X_3 + X_1 \cdot X_2 + X_1 \cdot X_3 + X_2 \cdot X_3 \]

- \( S_4 \) has no simple interpretation in terms of independence
  \[ S_4 \equiv \pi \left[ (\cdot) \prod_{i=1}^{K-1} \beta_{1i} x_i \right] \equiv \pi \left[ \prod_{i=1}^{K-1} \beta_{1i} x_i \right] \]
  For each levels of \( X_3 \), the reduced models of \( S_4 \) have different coefficients for the main effects of \( X_1 \) and \( X_2 \), but have the same coefficients for the interaction \( X_1 \cdot X_2 \).
  - e.g., \( I=J=2 \), same fitted odds-ratio between \( X_1 \) and \( X_2 \) for each category of \( X_3 \).

**Q:** What does uniform association mean? How to interpret the association? How does it connect with interaction terms?

- \( S_4 \) is not saturated \( \Rightarrow \) some degrees of freedoms left for goodness-of-fit test
  - A saturated model corresponds to a 3-way table with different association between, say \( X_1 \) and \( X_2 \), across \( K \) levels of \( X_3 \) whereas \( Y \sim \Gamma \) corresponds to a 3-way table with constant \( \pi \).
Q: how to examine whether $X_1, X_2, X_3$ in a 3-way table are mutually independent ($S_1$), jointly independent ($S_2$), conditionally independent ($S_3$), or uniformly associated ($S_4$), individually?

Ans: Perform deviance-based/Pearson’s $X^2$ goodness-of-fit (GoF) tests for $S_1, S_2, S_3, S_4$ (as $H_0$), respectively.

However, be careful of zero or small $y_{ijk}$ (rule of thumb: 20% of cells less than 5) in the table ⇒ there will be some doubt about the accuracy of chi-square approximation in GoF test.

The chi-square approximation is better in comparing models than assessing GoF.

Analysis strategy: start with complex Poisson GLM (e.g., saturated one) and see how far the model can be reduced (e.g., using model selection or sequential deviance-based tests as in ANOVA to compare models).

Binomial (or multinomial) GLM approach for 3-way table:

- If $y_{ijk}$’s regarded as fixed, can treat $Y_{X_3}$ as response and $X_1, X_2$ as covariates.

Q1: what information been gone? Q2: what still attainable?

Ans for Q1: information about $\pi_{ij+}$

Ans for Q2: information about $\pi_{k|ij}$

Q: how is a binomial GLM connected to a Poisson GLM in 3-way tables?

- $Y_{X_3} \sim Binomial(y_{ij+}, \pi_{k=1|ij})$ when $K=2$
- $Y_{X_3} \sim Multinomial(y_{ij+}, \pi_{k=1|ij}, ..., \pi_{k=K|ij})$ when $K > 2$

The binomial GLM implicitly assumes an association between $X_1$ and $X_2$ (Q: why?)

Poisson GLM allows us to drop the $X_1: X_2$ term, but binomial GLM does not.
Using binomial GLM loses little when we are interested in the relationship between the response \(X_3\) and the two covariates \(X_1, X_2\), and not interested in the association between \(X_1\) and \(X_2\).

\[Y_{X_3} \sim 1 + X_1 \iff X_2, \text{ given } X_1\]  
A function of \(X_1\) only.

\[Y_{X_3} \sim 1 + X_1 + X_2 \iff S_4 \text{ (uniform association)}\]

The saturated binomial GLM, \(Y_{X_3} \sim 1 + X_1 + X_2 + X_1: X_2\), corresponds to a Poisson GLM for different association.

Using binomial GLM loses little when we are interested in the relationship between the response \(X_3\) and the two covariates \(X_1, X_2\), and not interested in the association between \(X_1\) and \(X_2\).

Q: Poisson or binomial GLM approach? Which to use?

- Poisson if relationship between 3 variables is more symmetric

Q: How about \(Y_{X_3} \sim 1 + X_2\)?

Q: Can we exam whether \(X_1, X_2\) are independent given \(X_3\)?

- Correspondence analysis
  - Cannot directly apply to 3-way table
  - Can combine two of the factors, say \(X_1\) and \(X_2\), into a factor with \(I \times J\) levels and apply correspondence analysis on the 2-way table formed by the new factor and \(X_3\)
  - Q: which two factors should be chosen to merge?
    - Ans: pick up the two whose association is least interesting to us

\[\text{smoker: } \frac{109}{145} = .75\]
\[\text{smoker: } \frac{121}{250} = .48\]

Simpson’s paradox

- example:

<table>
<thead>
<tr>
<th>(X_1(i):) age</th>
<th>(X_2(j):) smoker</th>
<th>(X_3(k):) dead or alive</th>
<th>(c_{ij})</th>
</tr>
</thead>
<tbody>
<tr>
<td>age=35-44</td>
<td>yes</td>
<td>young</td>
<td>dead</td>
</tr>
<tr>
<td>yes</td>
<td>14</td>
<td>95</td>
<td>.13</td>
</tr>
<tr>
<td>no</td>
<td>7</td>
<td>114</td>
<td>.06</td>
</tr>
<tr>
<td>age=65-74</td>
<td>no</td>
<td>older</td>
<td>dead</td>
</tr>
<tr>
<td>yes</td>
<td>29</td>
<td>7</td>
<td>.81</td>
</tr>
<tr>
<td>no</td>
<td>101</td>
<td>28</td>
<td>.78</td>
</tr>
</tbody>
</table>

Higher proportion of young people in smoker

<table>
<thead>
<tr>
<th>smoker</th>
<th>dead</th>
<th>alive</th>
</tr>
</thead>
<tbody>
<tr>
<td>yes</td>
<td>43</td>
<td>102</td>
</tr>
<tr>
<td>no</td>
<td>108</td>
<td>142</td>
</tr>
</tbody>
</table>

\[\text{smoker: } \frac{145}{250} = .58\]

\[\text{smoker: } \frac{145}{250} = .57\]

made by S.-W. Cheng (NTHU, Taiwan)