Combinatorial Analysis

- An example:
  - A communication system is to consist of \( n \) seemingly identical antennas that are to be lined up in a linear order.
  - A resulting system will be functional as long as no two consecutive antennas are defective.
  - If it turns out \( m (=2) \) of the \( n (=4) \) antennas are defective, what is the probability that the resulting system will be functional?

- Many problems in probability theory can be solved simply by counting the number of different ways that a certain event can occur.
- The mathematical theory of counting is formally known as combinatorial analysis.


- Lists
  - Definition
    - Ordered Pairs: \((x, y) = (w, z)\) iff \( w = x \) and \( z = y \).
    - Ordered Triples: \((x, y, z) = (u, v, w)\) iff \( u = x \), \( v = y \), and \( w = z \).
    - List of Length \( r \): \((x_1, \ldots, x_r) = (y_1, \ldots, y_s)\) iff \( s = r \) and \( x_i = y_i \) for \( i = 1, \ldots, r \).
  - Example (License Plates): A license plate has the form \( LMNwxyz \), where
    - \( L, M, N \in \{A, B, \ldots, Z\} \),
    - \( w, x, y, z \in \{0, 1, \ldots, 9\} \),
    - and, so, is a list of length seven.
  - The basic principle of counting - multiplication principle
    - For two: If there are \( m \) choices for \( x \) and for each choice of \( x \), \( n \) choice for \( y \), then there are \( mn \) choices for \((x, y)\).
    - For several: If there are \( n_i \) choices for \( x_i \), \( i = 1, \ldots, r \), then there are \( n_1 n_2 \cdots n_r \) choices for \((x_1, \ldots, x_r)\).
Example:
As I was going to St. Ives, I met a man with seven wives
Every wife had seven sacks, Every sack had seven cats
Every cat had seven kits, Kits, cats, sacks, wives
How many were going to St. Ives?

- Ans: none
- However, how many were going the other way?
  7 Wives, 7×7=49 sacks, 49×7=343 cats, 343×7=2401 kits
  Total=7+49+343+2401=2800

Example (license plates): A license plate has the form
LMNwxyz, where
L, M, N ∈ {A, B,..., Z}
w, x, y, z ∈ {0, 1,..., 9}
There are $26^3 \times 10^4 = 175,760,000$ license plates. Of these,
$(26 \times 25 \times 24) \times (10 \times 9 \times 8 \times 7) = 78,624,000$
of them have distinct letters and digits (no repetition).

- Permutation ($r$-permutation of $n$ objects, $r \leq n$)
  - Definition: For $n$ objects, a permutation of length $r$ is a list
    $(x_1, ..., x_r)$ with distinct components (no repetition); that is
    $x_i \neq x_j$ when $i \neq j$
  - Example: (1, 2, 3) is a permutation of three elements; (1, 2, 1) is
    not a permutation
  - Counting Formulas. From $n$ objects, there are
    $n^r = n \times \cdots \times n$ (r factors)
    lists of length $r$ and
    $(n)_r = n \times (n-1) \times \cdots \times (n-r+1)$
    permutations of length $r$ may be formed.
  - Example: There are $10^3=1000$ three digit numbers, of which
    $(10)_3=10 \times 9 \times 8=720$ lists with distinct digits.
  - Some notations
    - Factorials: For positive integers $n$ and $r$, when $r=n$, write
      $n! = (n)_n = n \times (n-1) \times \cdots \times 2 \times 1$
    - Conventions: $(n)_0=1$ and $0!=1$
Some Notes
- The textbook only consider $r=n$.
- $(n)_r \equiv 0$, if $r>n$.
- If $r<n$, then $n! = (n)_r (n-r)!$

Example: A group of 9 people may choose officers (P, VP, S, T) in $(9)_4 = 3024$ ways.

Example:
- 7 books may be arranged in $7! = 5040$ ways
- If there are 4 math books and 3 science books, then there are $2 \times (4! \times 3!) = 288$ arrangements in which the math books are together and the science books are together.

• Combinations
  - Definition: For $n$ objects, a combination of size $r$ is a set \{ $x_1$, $\ldots$, $x_r$ \} of $r$ distinct elements. Two combinations equal if they have the same elements, possibly written in different order.

Example: \{1, 2, 3\} = \{3, 2, 1\}, but \{1, 2, 3\} \neq \{3, 2, 1\}

Example: How many committees of size 4 may be chosen from 9 people? Choose officers in two steps:
- Choose a committee in ?? Ways.
- Choose officers from the committee in 4! Ways.

From the Basic principle
- $(9)_4 = 4! \times ??$
- So, ?? = $(9)_4 / 4! = 126$

• Combinations Formula
- From $n \geq 1$ objects, \[ \binom{n}{r} = \frac{1}{r!} (n)_r \]
  combinations of size $r \leq n$ may be formed
- Example (bridge): A bridge hand is a combination of $r=13$ cards drawn from a standard deck of $n=52$. There are \[ \binom{52}{13} = 635, 013, 559, 600 \] such hands.
• Binomial coefficients

- Alternatively, \[ \binom{n}{r} = \frac{n!}{r!(n-r)!}. \]

- **The Binomial Theorem**: For all \(-\infty < x, y < \infty\)

\[ (x + y)^n = \sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r}. \]

  - **Proof.** If \((x + y)^n = (x + y) \times \cdots \times (x + y)\).
  
    is expanded, then \(x^r y^{n-r}\) will appear as often as \(x\) can be chosen from \(r\) of the \(n\) factors; i.e., in \(\binom{n}{r}\) ways

- **Example.** When \(n = 3\), \((x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3\).

- **Binomial identities**

  - Setting \(x = y = 1\), we get

    - Example: how many subsets are there of a set consisting of \(n\) elements?

  - Letting \(x = -1\) and \(y = 1\), we get

- **A useful identity:**

\[ \binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r} \]

- **Partitions**

- **Example**: How many distinct arrangements formed from the letters \(\text{M I S S I S S I P P I} \)?

  - There are 11 letters which can be arranged in \(11!\) Ways

  - But, this leads to double counting. If the 4 “S” are permuted, then nothing is changed. Similarly, for the 4 “I”s and 2 “P”s.

  - Each configuration of letters counted

\[ 4! \times 4! \times 2! = 1,152 \]

  times and the answer is \(\frac{11!}{4!4!2!} = 34,650\).

- **Definition:** Let \(Z\) be a set with \(n\) objects. If \(r \geq 2\) is an integer, then, an **ordered partition** of \(Z\) into \(r\) subsets is a list

\[ (Z_1, \ldots, Z_r) \]

where \(Z_1, \ldots, Z_r\) are mutually exclusive subsets of \(Z\) whose union is \(Z\); i.e.,
- \( Z_i \cap Z_j = \emptyset \), if \( i \neq j \), and
- \( Z_1 \cup \cdots \cup Z_r = Z \).

Let \( n_i = \#Z_i \), the number of elements in \( Z_i \). Then, \( n_1, \ldots, n_r \geq 0 \), and \( n_1 + \cdots + n_r = n \).

Example: In the “MISSISSIPPI” example, 11 positions, 
\[ Z=\{1, 2, \ldots, 11\} \]
were partitioned into four groups of size 
\( n_1 = 4 \) “I”s, \( n_2 = 1 \) “M”s, \( n_3 = 2 \) “P”s, \( n_4 = 4 \) “S”s.

In a bridge game, a deck of 52 cards is partitioned into four hands of size 13 each, one for each of South, West, North, and East.

The Partitions Formula. Let \( n, r \geq 1 \), and \( n_1, \ldots, n_r \geq 0 \) be integers s.t. \( n_1 + \cdots + n_r = n \). If \( Z \) is a set of \( n \) objects, then there are 
\[
\binom{n}{n_1, \ldots, n_r} = \frac{n!}{n_1! \times \cdots \times n_r!}
\]
(called multinomial coefficients) ways to partition \( Z \) into \( r \) subsets \( (Z_1, \ldots, Z_r) \) for which \( \#Z_i = n_i \) for \( i = 1, \ldots, r \).

The multinomial theorem
\[
(x_1 + \cdots + x_r)^n = \sum_{n_1+\cdots+n_r=n} \binom{n}{n_1, \ldots, n_r} x_1^{n_1} \cdots x_r^{n_r}.
\]

Examples:
- 9 children divided into A, B, C
  3 teams of 3 each. How many different divisions?
- 9 children divided into 3 groups of 3 each, to play a game. How many different divisions?
- a knockout tournament involving \( n = 2^m \) players
  - \( n \) players divided into \( n/2 \) pairs
  - losers of each pair eliminated; winner go next round
  - the process repeated until a single player remains
  - \( Q \): How many possible outcomes for the 1st round?
- \( Q \): How many possible outcomes of the tournament?
• The Number of Integer Solutions
  ➢ If \( n \) and \( r \) are positive integers, how many integer solutions are there to the equations: \( n_1, \ldots, n_r \geq 0 \) and \( n_1 + \cdots + n_r = n \)?
  ➢ Example: How many arrangements from \( a \) A's and \( b \) B's, for example, ABAAB? There are \( \binom{a + b}{a} = \binom{a + b}{b} \) such arrangements, since an arrangement is determined by the \( a \) places occupied by A.
  ➢ Example: Suppose \( n = 8 \) and \( r = 4 \). Represent solutions by “o” and “+” by “|”.
    - For example, \( ooo|oo||oooomean \( n_1 = 3, n_2 = 2, n_3 = 0, n_4 = 3 \).
    - Note: only \( r - 1 \) (=3) “|”s are needed.
    - There are as many solutions as there are ways to arrange “o” and “|”. By the last example, there are \( \binom{8 + 3}{3} = \binom{11}{3} = 165 \) solutions.
  ➢ A general formula. For positive integers \( n \) and \( r \), there are
    \[
    \binom{n + r - 1}{r - 1} = \binom{n + r - 1}{n} \]
    integer solutions to \( n_1, \ldots, n_r \geq 0 \) and \( n_1 + \cdots + n_r = n \).
  ➢ If \( n \geq r \), then there are \( \binom{n - 1}{r - 1} \) solutions with \( n_i \geq 1 \), for \( i = 1, \ldots, r \).

❄ Reading: textbook, Chapter 1