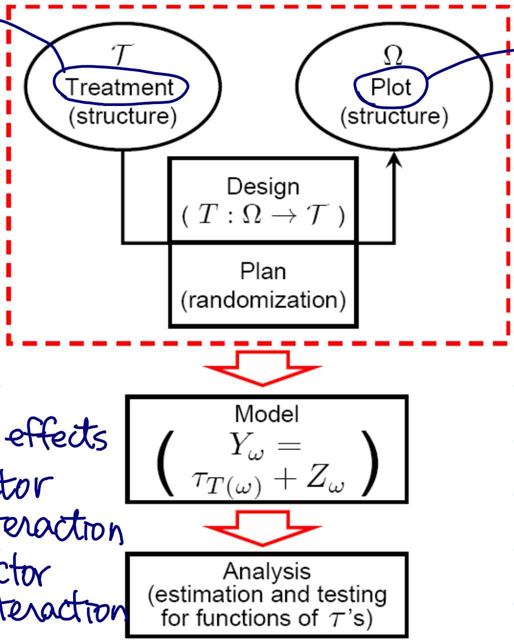
$\{1, 2, \cdots, t\}$

$2^k, 3^\ell, 4^k \cdot 6^\ell, \ldots$



N plots: generated by iterated crossing & nesting

$V = V_0 \oplus W_T \oplus V_T^{\perp}$

$W_T = W_T^{(1)} \oplus W_T^{(2)} \oplus \cdots \oplus W_T^{(r)}$

main effects, 2-factor interaction, 3-factor interaction

$V = S_0 \oplus S_1 \oplus \cdots \oplus S_p$

☆ $\boxed{\text{Q}: W_T^{(i)} \in \text{which stratum?}}$

Factorial experiments

n two-level factors

$x_i = 1, \ -1, \ i = 1, 2, \cdots, n$

$$f(x_1, x_2, \cdots, x_n) = \mu + \sum_{i=1}^{n} \beta_i x_i + \sum_{1 \le i < j \le n} \beta_{ij} x_i x_j$$
$$+ \sum_{1 \le i < j < k \le n} \beta_{ijk} x_i x_j x_k + \cdots + \beta_{12\cdots n} x_1 \cdots x_n$$

μ: mean, β_i: main effect, β_{ij}: two-factor interaction,...

full factorial: run size $= 2^n$

Fractional factorial designs

2^{n-p} regular fractional factorial designs

$n = \#$ of factors

$N = 2^{n-p} =$ run size

$\frac{1}{2^p}$-fraction of 2^n complete factorial

Fractional Factorial Designs (FFD)

	A	B	C	a	b	c	ab	ac	bc	abc
1	−1	−1	−1	−1	−1	−1	1	1	1	−1
2	−1	−1	1	−1	−1	1	1	−1	−1	1
3	−1	1	−1	−1	1	−1	−1	1	−1	1
4	−1	1	1	−1	1	1	−1	−1	1	−1
5	1	−1	−1	1	−1	−1	−1	−1	1	1
6	1	−1	1	1	−1	1	−1	1	−1	−1
7	1	1	−1	1	1	−1	1	−1	−1	−1
8	1	1	1	1	1	1	1	1	1	1

	A	B	C	a	b	c	ab	ac	bc	abc
2	−1	−1	1	−1	−1	1	1	−1	−1	1
3	−1	1	−1	−1	1	−1	−1	1	−1	1
5	1	−1	−1	1	−1	−1	−1	−1	1	1
8	1	1	1	1	1	1	1	1	1	1

- effect^2=I

  $a^2=b^2=c^2=(ab)^2=(ac)^2$

  $=(bc)^2=(abc)^2=I$
- defining relation

  $I = abc$
- effect aliasing

  a=bc

  b=ac

  c=ab
- un-aliased effects are orthogonal

- A design of size N can accommodate at most $N-1$ two-level factors
- A saturated design of size $N=2^k$ can be constructed by writing all possible combinations of k factors in a $2^k \times k$ array, and then completing all possible component-wise products of the columns.
- A regular design with n factors is obtained by choosing n columns from the saturated design.

If we use capital letters to denote the factors, then we also use combinations of these letters to denote interactions

A: main effect of factor A
AB: interaction of factors A and B
BCE: interaction of factors B, C and E
etc.

$$x_4 = x_1x_2, \; x_5 = x_1x_3$$

$$1 = x_1x_2x_4 = x_1x_3x_5 = x_2x_3x_4x_5$$

Defining relation

$$I = ABD = ACE = BCDE$$

Defining words

Defining contrasts, Defining effects

$$1 = x_1x_2x_4 = x_1x_3x_5 = x_2x_3x_4x_5$$

$$x_1 = x_2x_4 = x_3x_5 = x_1x_2x_3x_4x_5$$

$$A = BD = CE = ABCDE$$

In the model matrix for the full model, the columns corresponding to the main effect A and interactions BD, CE, ABCDE are identical. Therefore they are completely mixed up. We say they are aliases of one another.

$$\{A,\, BD,\, CE,\, ABCDE\}: \textit{alias set}$$

One can estimate only one effect in each alias set, assuming that all the other effects in the same alias set are negligible.

7 alias sets

If E is a plot factor, we would
- remove interaction between plot & treatment
- E=AC=BCD is called confounding.

$$A = BD = \cancel{CE} = \cancel{ABCDE}$$
$$B = AD = \cancel{ABCE} = \cancel{CDE}$$
$$C = ABCD = \cancel{AE} = \cancel{BDE}$$
$$D = AB = \cancel{ACDE} = \cancel{BCE}$$
$$\boxed{E} = \cancel{ABDE} = AC = BCD$$
$$BC = ACD = \cancel{ABE} = \cancel{DE}$$
$$CD = ABC = \cancel{ADE} = \cancel{BE}$$

In general, among the $2^n - 1$ factorial effects, $2^p - 1$ appear in the defining relation. The rest are divided into $2^{n-p} - 1$ alias sets, each of size 2^p.

Regular fractional factorial designs have simple alias structures: any two factorial effects are either orthogonal or completely aliased.

Nonregular designs have complex alias structures that are difficult to disentangle.

Under the design defined by I = ABD = ACE = BCDE, the usual estimate of the main effect of A actually estimates A + BD + CE + ABCDE. This is an unbiased estimate of A if all its aliases are negligible.

joint effect.

When some contrasts are found significant but cannot be attributed to specific effects, one has to perform follow-up experiments to resolve the ambiguity. This is called de-aliasing.

Which of the following two 2^{7-2} resolution IV designs is better?

d_1: $I = DEFG = ABCDF = ABCEG$

d_2: $I = ABCF = ADEG = BCDEFG$

Minimum aberration (Fries and Hunter, 1980): Technometrics

Sequentially minimize $A_1, A_2, \cdots$, where A_i = number of words of length i in the defining relation

Word length pattern $(A_1, A_2, \cdots)$

Design Key

- Patterson (1965) *J. Agric. Sci.*
- Patterson (1976) *JRSS, Ser. B*
- Bailey, Gilchrist and Patterson (1977) *Biometrika*
- Bailey (1977) *Biometrika*
- Patterson and Bailey (1978) Applied Statistics

Factorial design construction, identification of effect aliasing and confounding with block factors

- notation & definition

$T_1, \cdots, T_m$: m treatment factors, each has t_i levels

$P_1, \cdots, P_n$: n plot factors (block factor, row/column factor, nested factor)

each has n_j levels.

- pseudo-factor: for a factor with $p_1^{v_1} p_2^{v_2} \cdots p_r^{v_r}$ levels, where $p_1, \cdots, p_r$ are different prime numbers, we can use v_1 p_1-level, v_2 p_2-level, $\cdots$, v_r p_r-level pseudo-factors to represent it. e.g.

4-level = 2^2-level

X	X_1	X_2
0	0	0
1	0	1
2	1	0
3	1	1

X : 3 main effects

$\downarrow$

$\boxed{X_1, X_2, X_1X_2}$

- When a particular treatment effect is confounded with a plot effect, the plot effect is called plot alias.
- A design key specifies the plot aliases of the treatment main effects

Situation 1: Symmetric $\Rightarrow$ # of levels for factors are p^{m_i}, p: a prime number.

a. single replicate (i.e., $N=t$)

(i) full factorial design for treatment

(ii) fractional factorial design for treatment

b. multiple replicates (i.e. $N=2t, 3t, \cdots$)

(i) full factorial for treatment

(ii) fractional factorial for treatment

situation 2: asymmetric $\Rightarrow$ # of levels $= p_1^{m_1} \cdot \cdots \cdot p_r^{m_r}$

Q: how to find $W_T^{(i)} \in$ which stratum?

step 1. identify strata

step 2. use pseudo-factors to represent the factors

step 3. decompose each stratum into plot effects represented by the plot pseudo-factors.

step 4. choose a design key for the plot aliases of treatment main effects represented by pseudo-factors

step 5. use design key to find the plot aliases of other treatment effects.

$N=t$ A Graeco-Latin square

Treatment structure

$A \times B$
5 5

full factorial

~~Block~~ plot structure

$U \times V$ — plot effects
5 5

	df
$AB^2 = U$	4
V	4
$A = UV$	4
$B = UV^2$	4
UV^3	4
$AB = UV^4$	4

Design key

A	B
UV	UV^2

Recall: LNp.5-32 ← **Stratum identification**

Treatment effect	d.f.	Plot alias	Stratum
A	4	UV	UV
B	4	UV^2	UV
AB	4	U^2V^3	UV
AB^2	4	U^3	U
AB^3	4	U^4V^2	UV
AB^4	4	V^4	V

S_E

dim=25

$\mathcal{V}$

treatment structure		plot structure	dim	
	V_0	S_0	1	
A, B, AB, AB^2, AB^3 (AB^4)	$\oplus$ W_T	$\oplus$ S_r	4	U
	$\oplus$ $\cancel{W_T^{\perp}}$	$\oplus$ S_c	4	V
		S_E	16	UV, UV^2, UV^3, UV^4 (df=4×4)

Inverse key

U	V
A^2B^4	A^4B

Rules of construction

$$q(A) = q(U) + q(V) \mod 5$$
$$q(B) = q(U) + 2q(V) \mod 5$$

Construction of design

		Column (level of V)				
		0	1	2	3	4
		Levels of A (first) and B (second)				
Row (level of U)	0	00	12	24	31	43
	1	11	23	30	42	04
	2	22	34	41	03	10
	3	33	40	02	14	21
	4	44	01	13	20	32

$$AB = (UV)(UV^2) = U^2V^3 \approx UV^4$$
$$\llcorner U^6V^9 \lrcorner \text{(mod 5)}$$

$$AB^2 = UV \cdot U^2V^4$$
$$= U^3V^5 = U^3 \approx U$$

$$AB^3 = \cdots$$

$$AB^4 = \cdots$$

A single replicate block design

$(N=t)$

Treatment structure

$\boxed{\begin{matrix} A \times B \times C \times D \\ 2 \quad 2 \quad 2 \quad 2 \end{matrix}}$

full factorial

~~Block~~ plot structure (nested factor)

block factor — X / Y
4 4
X_1 X_2 Y_1 Y_2

Design key

A	B	C	D
Y_1	Y_2	$X_1Y_1Y_2$	$X_2Y_1Y_2$

Inverse key

X_1	X_2	Y_1	Y_2
ABC	ABD	A	B

Stratum contents

Stratum	d.f.	Treatment effects
X	3	ABC, ABD, CD
XY	12	The rest

Partial inverse key

X_1	X_2
ABC	ABD

Rules of construction

$$q(A) = q(Y_1) \mod 2$$
$$q(B) = q(Y_2) \mod 2$$
$$q(C) = q(X_1) + q(Y_1) + q(Y_2) \mod 2$$
$$q(D) = q(X_2) + q(Y_1) + q(Y_2) \mod 2$$

$\mathcal{V}$ — plot structure

			dim	
	V_0	S_0	1	
A, B, C, D, AB, AC, …, $ABCD$	$\oplus$ W_T	$\oplus$ S_B	3	X_1, X_2, X_1X_2
	$\oplus$ $\cancel{W_T^{\perp}}$	$\oplus$ S_E	12	$Y_1 = A$, $Y_2 = B$, Y_1Y_2; X_1Y_1, X_1Y_2, $X_1Y_1Y_2 = C$; X_2Y_1, X_2Y_2, $X_2Y_1Y_2 = D$; $X_1X_2Y_1$, $X_1X_2Y_2$, $X_1X_2Y_1Y_2$

$AB = Y_1Y_2$
$AC = X_1Y_2$
⋮
$ABC = Y_1Y_2X_1Y_1Y_2 = X_1$
⋮
$ABD = X_2$
⋮
$CD = X_1X_2$
⋮

★ assume some higher-order interactions are negligible, some d.f. are left for testing

main effects A, B confounded with plot main effects

design matrix

A	B	C	D	X_1 ($=ABC$)	X_2 ($=ABD$)	→ X
0	0	0	0	0	0	0
0	0	0	1	0	1	1
0	0	1	0	1	0	2
0	0	1	1	1	1	3
0	1	0	0	1	1	3
0	1	0	1	1	0	2
0	1	1	0	0	1	1
0	1	1	1	0	0	0
1	0	0	0	1	1	3
1	0	0	1	1	0	2
1	0	1	0	0	1	1
1	0	1	1	0	0	0
1	1	0	0	0	0	0
1	1	0	1	0	1	1
1	1	1	0	1	0	2
1	1	1	1	1	1	3

↳ a list of treatment effects whose plot aliases are plot main effects

(N=t)

A design with crossing and nesting

Treatment structure

$A \times B \times C$
3 3 3

full factorial

~~Block~~ plot **structure**

$(U \times V)/W$
3 3 3

Design key

A	B	C
W	UVW	UV^2W

U, V, UV, UV^2, W, UW, VW, UVW, UV^2W, UW^2, VW^2, UVW^2, UV^2W^2

Stratum identification

Treatment effect	d.f.	Plot alias	Stratum
A	2	W	UVW
B	2	UVW	UVW
C	2	UV^2W	UVW
AB	2	UVW^2	UVW
AB^2	2	U^2V^2	UV
AC	2	UV^2W^2	UVW
AC^2	2	U^2V	UV
BC	2	U^2W^2	UVW
BC^2	2	V^2	V
ABC	2	U^2	U
ABC^2	2	V^2W	UVW
AB^2C	2	VW	UVW
AB^2C^2	2	UW^2	UVW

$\in S_4$

Inverse key

U	V	W
$A^2B^2C^2$	B^2C	A

Partial inverse key → design matrix

U	V
$A^2B^2C^2$	B^2C

$N(c(n_1,n_2),n_3)$
$= 1 + (c(n_1,n_2)-1) + \cancel{c(n_1,n_2)}\, n_1n_2\,(n_3-1)$

$\quad c(n_1,n_2) = 1+(n_1-1)(n_2-1)+(n_1-1)(n_2-1)$

$= \underbrace{1}_{S_0} + \underbrace{(n_1-1)}_{S_1} + \underbrace{(n_2-1)}_{S_2} + \underbrace{(n_1-1)(n_2-1)}_{S_3} + \underbrace{[n_1n_2 - \cancel{\quad}](n_3-1)}_{S_4}$

$\mathcal{V}$ — plot structure

	dim	
S_0	1	
$\oplus$		
S_1	2	— U
$\oplus$		
S_2	2	— V
$\oplus$		
S_3	4	— UV, UV^2
$\oplus$		
S_4	18	— other plot effects with W.

$AB = W \cdot UVW = UVW^2 \in S_4$
⋮

(N=t)

A half-replicate design

full factorial

Treatment structure

$A \times B \times C \times D \times E$
2 2 2 2 2

full: 32 run
$\frac{1}{2}$ FFD: 16 run

~~Block~~ plot **structure**

X/Y
2 8

→ # of plots = 16

Y_1 Y_2 Y_3

Design key

A	B	C	D	E
Y_1	Y_2	Y_3	XY_1Y_2	$Y_1Y_2Y_3$

temperaly ignore

32 all treatment effects.

$A = BCE$
$B = ACE$
$C =$
D
E
AB
AC
⋮
$ABCDE$

Inverse key

X	Y_1	Y_2	Y_3	I
ABD	A	B	C	$ABCE$

Partial inverse key

X	I
ABD	$ABCE$

recover later

$BCE = A = Y_1$
$ACE = B = Y_2$
$\quad = C = Y_3$
$D = XY_1Y_2$
$AB = Y_1Y_2$
$AC = \cdots$
$\boxed{CDE = ABD = X}$ ← fixed effect approach, only the alias set $ABD = CDE$ are sacrified for block effects.
$ABCD =$

$\mathcal{V}$

		dim	
V_0	S_0	1	
$\oplus$	$\oplus$		
W_T	S_B	1	— X
$\oplus$	$\oplus$		
~~W_E~~	S_E	14	— Y_1, Y_2, Y_3, Y_1Y_2, Y_1Y_3, Y_2Y_3, $Y_1Y_2Y_3$; XY_1, XY_2, ⋮

☆ Split-plot design
the assignment of design key is restricted