Lemma: If $\boldsymbol{a}'\boldsymbol{y}$ is an unbiased estimator of a treatment contrast $\boldsymbol{c}'\boldsymbol{\alpha}$, then $\boldsymbol{a}'\mathbf{1}_N = 0$, i.e., $\boldsymbol{a} \in \mathcal{G}^\perp$ $\quad \mathcal{G} = \text{Span}(\mathbb{1})$

$c'\mathbb{1} = 0$

Proof: $E(\boldsymbol{a}'\boldsymbol{y}) = \mu\boldsymbol{a}'\mathbf{1} + \boldsymbol{a}'\boldsymbol{X}_T\boldsymbol{\alpha} \overset{\text{unbiased.}}{=} \boldsymbol{c}'\boldsymbol{\alpha}$ for all μ and $\boldsymbol{\alpha}$

$$\Rightarrow \boldsymbol{a}'\boldsymbol{X}_T = \boldsymbol{c}' \Rightarrow \boldsymbol{a}'\boldsymbol{X}_T\mathbf{1}_t = \boldsymbol{c}'\mathbf{1}_t = 0 \Rightarrow \boldsymbol{a}'\mathbf{1}_N = 0. \quad \square$$

Hence

$\begin{bmatrix} \bar{y} \\ \vdots \\ \bar{y} \end{bmatrix}$ projection matrix of $\mathcal{G}$

$$\boldsymbol{a}'\boldsymbol{y} = \boldsymbol{a}'(\boldsymbol{y} - \boldsymbol{G}\boldsymbol{y}) + \underbrace{\boldsymbol{a}'\boldsymbol{G}\boldsymbol{y}}_{=0} = \boldsymbol{a}'(\boldsymbol{y} - \boldsymbol{G}\boldsymbol{y})$$

All linear unbiased estimators of a treatment contrast are functions of $\boldsymbol{y} - \boldsymbol{G}\boldsymbol{y}$.

> Projections of the data vector onto different strata are uncorrelated between and homoscedastic within.

$\hookrightarrow \mathcal{S}_0, \mathcal{S}_1, \ldots, \mathcal{S}_s$

$Y = X\beta + \varepsilon,\ \text{cov}(\varepsilon) = \boxed{\sigma^2 I}$

error structures have same variance

Project the randomization model onto $\mathcal{S}_i$, $i \geq 1$:

$E(y) = \mu + X_T\alpha$

$\text{cov}(y) = V$

$y = \mu + X_T\alpha + \varepsilon$

$$\overset{y^{(i)}}{\boldsymbol{P}_i\boldsymbol{y}} = \overset{X^{(i)}}{\boldsymbol{P}_i\boldsymbol{X}_T}\boldsymbol{\alpha} + \overset{\varepsilon^{(i)}}{\boldsymbol{P}_i\boldsymbol{\epsilon}}$$

$\text{cov}(P_i\varepsilon) = P_i\,\text{cov}(\varepsilon)P_i^T = P_i V P_i^T = P_i\left(\sum_{k=0}^{s}\xi_k P_k\right)P_i^T = \xi_i P_i P_i P_i^T = \xi_i P_i$

$$\text{E}(\boldsymbol{P}_i\boldsymbol{\epsilon}) = \mathbf{0} \text{ and } \text{cov}(\boldsymbol{P}_i\boldsymbol{\epsilon}) = \xi_i\boldsymbol{P}_i,$$

When restricted to the vectors in $\mathcal{S}_i$, the covariance matrix $\xi_i\boldsymbol{P}_i$ is the same as $\xi_i\boldsymbol{I}$: $(\xi_i\boldsymbol{P}_i)\boldsymbol{v} = \xi_i\boldsymbol{v}$ for all $\boldsymbol{v} \in \mathcal{S}_i$. Therefore the Gauss-Markov Theorem applies, and the least squares estimators are the best linear unbiased estimators.

homoscedastic within

LS estimate of α in ith stratum

uncorrelated between $\quad i \neq j \Rightarrow \text{cov}(\boldsymbol{P}_i\boldsymbol{y}, \boldsymbol{P}_j\boldsymbol{y}) = 0$

$(X^{(i)T}X^{(i)})\alpha = (X^{(i)})' \cdot Y^{(i)}$

Normal equation $$[(\boldsymbol{P}_i\boldsymbol{X}_T)'\boldsymbol{P}_i\boldsymbol{X}_T]\hat{\boldsymbol{\alpha}}^i = (\boldsymbol{P}_i\boldsymbol{X}_T)'\boldsymbol{P}_i\boldsymbol{y}$$

i.e., $$(\boldsymbol{X}_T'\boldsymbol{P}_i\boldsymbol{X}_T)\hat{\boldsymbol{\alpha}}^i = \boldsymbol{X}_T'\boldsymbol{P}_i\boldsymbol{y}$$

> Estimates computed in different strata are uncorrelated.
>
> Estimate each treatment contrast in each of the strata in which it is estimable, and combine the uncorrelated estimates from different strata.
>
> Simple analysis results when the treatment contrasts are estimable in only one stratum.

Lemma: A treatment function $\boldsymbol{c}'\boldsymbol{\alpha}$ is estimable in stratum $\mathcal{S}_i$ (i.e., $\exists$ an unbiased estimator of the form $\boldsymbol{a}'\boldsymbol{P}_i\boldsymbol{y}$) $\Leftrightarrow$ $\boldsymbol{c} \in \mathcal{R}(\boldsymbol{X}_T'\boldsymbol{P}_i\boldsymbol{X}_T)$.

Recall: If $R(X_T) \subset S_i$ $\Rightarrow R(X_T) \perp S_j$, $j \neq i$ $\Rightarrow P_j X_T = 0$ $\Rightarrow$ cannot estimate α in jth stratum

If $\boldsymbol{c}'\boldsymbol{\alpha}$ is estimable in stratum $\mathcal{S}_i$, then its BLUE in $\mathcal{S}_i$ is $\boldsymbol{c}'\hat{\boldsymbol{\alpha}}^i$, where $\hat{\boldsymbol{\alpha}}^i$ is any solution of the normal equation

$$\boldsymbol{X}_T'\boldsymbol{P}_i\boldsymbol{X}_T\hat{\boldsymbol{\alpha}}^i = \boldsymbol{X}_T'\boldsymbol{P}_i\boldsymbol{y},$$

and $\mathrm{var}(\boldsymbol{c}'\hat{\boldsymbol{\alpha}}^i) = \xi_i\, \boldsymbol{c}'(\boldsymbol{X}_T'\boldsymbol{P}_i\boldsymbol{X}_T)^-\boldsymbol{c}$.

$^-$: generalized inverse

$\boldsymbol{X}_T'\boldsymbol{P}_i\boldsymbol{X}_T$: *information matrix* for treatment effects in stratum $\mathcal{S}_i$

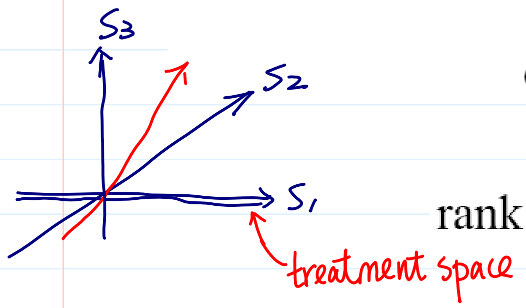
$\boldsymbol{X}_T'\boldsymbol{P}_i\boldsymbol{X}_T$ is symmetric, nonnegative definite and has zero row sums.

$\boldsymbol{c}'\boldsymbol{\alpha}$ is estimable $\Rightarrow$ it is a treatment contrast

Let

$$\boldsymbol{C}_i = \boldsymbol{X}_T'\boldsymbol{P}_i\boldsymbol{X}_T, \quad \boldsymbol{Q}_i = \boldsymbol{X}_T'\boldsymbol{P}_i\boldsymbol{y}.$$

Then the normal equation in stratum $\mathcal{S}_i$ becomes



$$\boldsymbol{C}_i\hat{\boldsymbol{\alpha}}^i = \boldsymbol{Q}_i.$$

$$\mathrm{rank}(\boldsymbol{C}_i) \le t - 1$$

If $\mathrm{rank}(\boldsymbol{C}_i) = t-1$, then we say that the design is *connected* in stratum $\mathcal{S}_i$. In this case, all the treatment contrasts are estimable in $\mathcal{S}_i$. i.e. any $c'\alpha$

$$\mathrm{E}(\boldsymbol{y}) = \boldsymbol{X}_T\boldsymbol{\alpha} \in \underbrace{\mathcal{R}(\boldsymbol{X}_T) = \mathcal{T}}_{\text{treatment space}}$$

$\boldsymbol{y}$: $(y - Gy)$

$$\mathrm{E}(\boldsymbol{P}_i\boldsymbol{y}) = \underbrace{\boldsymbol{P}_i\mathrm{E}(\boldsymbol{y})}_{P_i X_T \alpha} \in \boldsymbol{P}_i(\mathcal{T})$$

$\mathcal{T} = P_i \mathcal{T}$ if $\mathcal{T} \subset S_i$

$$\boldsymbol{P}_i\boldsymbol{y} \in \mathcal{S}_i$$

$$\underbrace{\mathcal{S}_i = \boldsymbol{P}_i(\mathcal{T}) \oplus (\mathcal{S}_i \ominus \boldsymbol{P}_i(\mathcal{T}))}_{\text{decompose into treatment space \& "residual" space.}}$$

Let $\boldsymbol{P}_{\boldsymbol{P}_i(\mathcal{T})}\boldsymbol{P}_i\boldsymbol{y}$ be the orthogonal projection of $\boldsymbol{P}_i\boldsymbol{y}$ onto $\boldsymbol{P}_i(\mathcal{T})$. Then

SS due to treatment

$$\|\boldsymbol{P}_i\boldsymbol{y}\|^2 = \|\boldsymbol{P}_{\boldsymbol{P}_i(\mathcal{T})}\boldsymbol{P}_i\boldsymbol{y}\|^2 + \underbrace{\|\boldsymbol{P}_{\mathcal{S}_i\ominus\boldsymbol{P}_i(\mathcal{T})}\boldsymbol{P}_i\boldsymbol{y}\|^2}_{\text{SS due to some block variation}}$$

ANOVA is stratum $\mathcal{S}_i$

Follow Thm2.4 in Bailey.

$$\|\boldsymbol{P}_{\boldsymbol{P}_i(\mathcal{T})}\boldsymbol{P}_i\boldsymbol{y}\|^2$$

$$= (\hat{\boldsymbol{\alpha}}^i)'\boldsymbol{Q}_i = (\hat{\boldsymbol{\alpha}}^i)'\boldsymbol{C}_i\hat{\boldsymbol{\alpha}}^i$$

$$\mathrm{E}\left(\tfrac{1}{dim\,(\mathcal{S}_i) - dim[\boldsymbol{P}_i(\mathcal{T})]}\|\boldsymbol{P}_{\mathcal{S}_i\ominus\boldsymbol{P}_i(\mathcal{T})}\boldsymbol{P}_i\boldsymbol{y}\|^2\right) = \xi_i$$

$$\mathrm{E}\left(\tfrac{1}{dim[\boldsymbol{P}_i(\mathcal{T})]}\|\boldsymbol{P}_{\boldsymbol{P}_i(\mathcal{T})}\boldsymbol{P}_i\boldsymbol{y}\|^2\right)$$

$$= \xi_i + \tfrac{1}{dim[\boldsymbol{P}_i(\mathcal{T})]}\boldsymbol{\alpha}'\boldsymbol{C}_i\boldsymbol{\alpha}$$

ANOVA within stratum $\mathcal{S}_i$:

Sources	Sums of Squares	d.f.	MS	E(MS)
Treatments	$(\hat{\alpha}^i)' C_i \hat{\alpha}^i$	$\dim[P_i(\mathcal{T})]$	$\frac{1}{\dim[P_i(\mathcal{T})]}(\hat{\alpha}^i)' C_i \hat{\alpha}^i$	$\xi_i + \frac{1}{\dim[P_i(\mathcal{T})]}\alpha' C_i \alpha$
Rresidual	By subtraction		$\cdots$	ξ_i
Total	$\Vert P_i y \Vert^2$	$\dim(\mathcal{S}_i)$		

$$\text{rank}(C_i) = \underline{\text{rank}(X_T' P_i X_T)} = \dim[P_i(\mathcal{T})]$$

$\Vert$ column rank of $X^{(i)}$

If the design is connected in $\mathcal{S}_i$, then the treatment sum of squares has $t-1$ degrees of freedom in $\mathcal{S}_i$.

ANOVA for a general randomized block design:

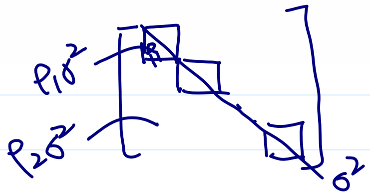
↳ nesting block structure.

	Sources of variation	SS	d.f.	MS	E(MS)
V_0					
$W_B \leftarrow$	Interblock				
	Treatments	$(\hat{\alpha}^1)' C_1 \hat{\alpha}^1$	rank (C_1)	$\cdots$	$\xi_1 + \frac{1}{\text{rank }(C_1)}\alpha' C_1 \alpha$
	Residual	By subtraction	$\vdots$		ξ_1
		$\sum_{i=1}^{b} k(y_{i\cdot} - y_{\cdot\cdot})^2$	$b-1$		
$W_E \leftarrow$	Intrablock				
	Treatments	$(\hat{\alpha}^2)' C_2 \hat{\alpha}^2$	rank (C_2)	$\cdots$	$\xi_2 + \frac{1}{\text{rank }(C_2)}\alpha' C_2 \alpha$
	Rresidual	By subtraction	$\vdots$		ξ_2
		$\sum_{i=1}^{b}\sum_{j=1}^{k}(y_{ij} - y_{i\cdot})^2$	$b(k-1)$		
	Total	$\Vert y - Gy \Vert^2$	$bk-1$		

$\rightarrow \because$ treatment space does not belong to any single stratum.

usually assume $\xi_1 > \xi_2$

$\text{cov}(y) =$



3 treatments, 3 blocks of size 3

1	1	1
2	2	2
3	3	3

All information for the treatment contrasts is contained in the intrablock stratum

1	2	3
1	2	3
1	2	3

treatment factor confounded with block factor

All information for the treatment contrasts is contained in the interblock stratum

1	2	2
1	3	3
2	1	3

Has tretament information In both inter- and intra-block strata

- same plot structure 3/3
- same treatment structure $\mathcal{T}=\{1,2,3\}$
- different design
- same plan

to compare the 1-3 difference within block, can only 2nd block (c.f. the 1st design)

ANOVA table

	d.f		
$\mathcal{V}_0$	1		
$\boxed{\mathcal{W}_B \cup \mathcal{W}_T}$	2	treatment	2
		residual	0
$\mathcal{W}_E$	7		

1	2	3
1	2	3
1	2	3

cannot do test

- but can estimate $\hat{\alpha}$ ($\because$ take random-effect model)

Q: $\hat{\alpha}$ should not be estimable because treatment factor confounded with block factor, correct?

- cannot estimate α
- but can do test. ($\because$ take fixed-effect model).

$$\underbrace{\mathcal{B}}_{\text{block space}} = \overbrace{\mathcal{T}}^{\text{treatment space}}$$

$$E(P_1\boldsymbol{y}) \in P_1(\mathcal{T}) = P_1(\mathcal{T} \ominus \mathcal{G}) = P_{\mathcal{B}\ominus\mathcal{G}}(\mathcal{T} \ominus \mathcal{G}) = \mathcal{T} \ominus \mathcal{G}$$

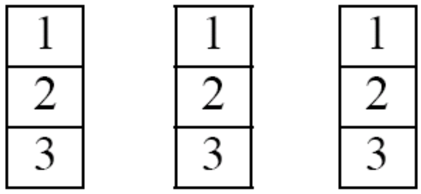
$$E(P_2\boldsymbol{y}) \in P_2(\mathcal{T}) = P_2(\mathcal{T} \ominus \mathcal{G}) = P_{\mathcal{B}^\perp}(\mathcal{T} \ominus \mathcal{G}) = \{0\}$$

The best linear unbiased estimator of a treatment contrast $\sum_{i=1}^t c_i\alpha_i$ is $\sum_{i=1}^t c_i\widehat{\alpha}_i$, where $\widehat{\alpha}_i$ is the ith treatment mean, with variance equal to $\frac{\xi_1}{3}\sum_{i=1}^t c_i^2$.

fixed effect approach	random effect approach
$y = \mu + X_T\alpha + X_B\beta + \varepsilon$	$y = \mu + X_T\alpha + X_B\beta + \varepsilon$
$E(y) = \mu + X_T\alpha + X_B\beta$	$E(y) = \mu + X_T\alpha$
$cov(y) = \sigma^2 I$	$cov(y) = V$

ANOVA

V_0	1
W_B	2
$W_E \cup W_T$	7 — 2 → treatment; 5 → residual



$$\mathcal{T} \ominus \mathcal{G} \perp \mathcal{B} \ominus \mathcal{G}$$

$$E(\boldsymbol{P}_1\boldsymbol{y}) \in \boldsymbol{P}_{\mathcal{B}\ominus\mathcal{G}}(\mathcal{T} \ominus \mathcal{G}) = \{0\}$$

$$E(\boldsymbol{P}_2\boldsymbol{y}) \in \boldsymbol{P}_{\mathcal{B}^\perp}(\mathcal{T} \ominus \mathcal{G}) = \mathcal{T} \ominus \mathcal{G}$$

The best linear unbiased estimator of a treatment contrast $\sum_{i=1}^t c_i\alpha_i$ is $\sum_{i=1}^t c_i\widehat{\alpha}_i = \sum_{i=1}^t c_i(i\text{th treatment mean}),$ with variance equal to $\frac{\xi_2}{3}\sum_{i=1}^t c_i^2$

estimation in stratum with ~~sm~~ smaller eigenvalue is more accurate.

In general, there may be information for treatment contrasts in more than one stratum. ($\because$ treatment space not included in a single stratum)

Analysis is still simple if the space of treatment contrasts $\mathcal{T} \ominus \mathcal{G}$ can be decomposed as $\oplus\, \mathcal{T}_i$, where each $\mathcal{T}_i$, consisting of treatment contrasts of interest, is entirely in one stratum.

design: Full factorial design 2^n

$\mathcal{T} = \{1, 2, 3, \cdots, 2^n\}$

$\dim(W_T) = 2^n - 1$

Orthogonal designs

main effects → $\mathcal{T}_1$
2-factor interaction → $\mathcal{T}_2$
3- . . . → $\mathcal{T}_3$
. . . → $\mathcal{T}_i$

$\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3, \mathcal{T}_i = \mathcal{T} \ominus \mathcal{G}$