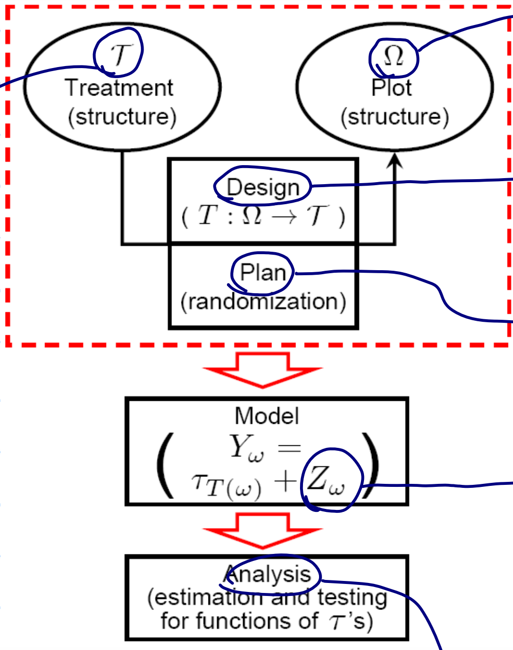
$\{1, \dots, t\}$



iterated nesting & crossing

best design
for nesting, RCBD
for crossing, row-column design

restricted randomization
- for nesting
- for crossing

fixed-effect model
random-effect model
randomization model

same $E(\mathbf{Y})$ & $cov(\mathbf{Y})$

within stratum comparison

Suppose each of N plots is assigned one of the f levels of a certain factor F. The relation between the plots and the levels of F can be described by an $N \times f$ incidence matrix

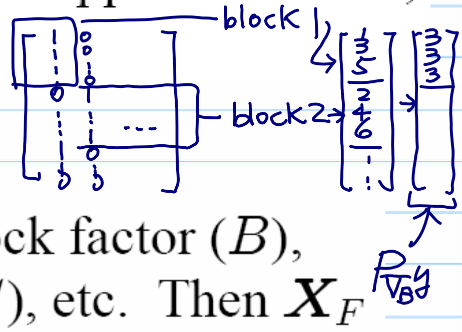
treatment factor
block :
row :
column :

$$\boldsymbol{X}_F = [x_{ij}],$$

where

$$x_{ij} = \begin{cases} 1, & \text{if the } j\text{th level of } F \text{ appears at } \text{~~unit~~ plot } i, \\ 0, & \text{otherwise} \end{cases}$$

$X_T = [u_1 | u_2 | \cdots | u_t]$ $\quad X_B = [v_1 | \cdots | v_b] =$



F can be a treatment factor (T), block factor (B), row factor (R), or column factor (C), etc. Then $\boldsymbol{X}_F$ is denoted $\boldsymbol{X}_T$, $\boldsymbol{X}_B$, $\boldsymbol{X}_R$, $\boldsymbol{X}_C$, respectively, and the number of levels is denoted t, b, r or c.

For any matrix $\boldsymbol{X}$, let $\mathcal{R}(\boldsymbol{X})$ be the range (or column space) of $\boldsymbol{X}$, i.e., the space generated by the column vectors of $\boldsymbol{X}$. $F=\mathcal{B} \Rightarrow R(X_F)=V_B \supseteq \mathcal{G}_B$

$R(X_T)=V_T$, $R(X_R)=V_R$, $R(X_C)=V_C$

Denote $\mathcal{R}(\boldsymbol{X}_F)$ by $\mathcal{F}$($\mathcal{B}$, $\mathcal{T}$, $\mathcal{R}$, or $\mathcal{C}$ for block, treatment, row or column factors, respectively).

$\mathcal{F} = \{\boldsymbol{y}$: $y_i = y_j$ if the same level of F appears at units i and $j\}$. If all the levels appear at least once, then dim $\mathcal{F} = f$. $\dim(V_B)=b$ $\dim(V_R)=r$ $\dim(V_T)=t$ $\dim(V_C)=c$

$$X_B = \begin{bmatrix} 1 & 0 & \\ 1 & 0 & \\ 1 & 0 & \\ 0 & 1 & \cdots \\ \vdots & 1 & \\ \vdots & 1 & \\ 0 & 0 & \end{bmatrix}$$

Let $\boldsymbol{F}$ be the orthogonal projection matrix onto $\mathcal{F}$. Then for any $\boldsymbol{y} \in R^N$ (V), $\boldsymbol{Fy}$ replaces each y_i with the average of the y_j's over all the units j which are assigned the same level of F as unit i.

$$B = P_{V_B} = \begin{bmatrix} \frac{1}{b_k}\begin{bmatrix} 1 & \cdots & 1 \\ 1 & \cdots & 1 \end{bmatrix} & & 0 \\ & \square & \\ 0 & & \ddots \ \square \end{bmatrix}$$

The orthogonal projection matrices onto $\mathcal{T}$, $\mathcal{B}$, $\mathcal{R}$ and $\mathcal{C}$ are denoted by $\boldsymbol{T}$, $\boldsymbol{B}$, $\boldsymbol{R}$ and $\boldsymbol{C}$, respectively.

P_{V_T} $P_{V_B} \leftrightarrow J_B$

Let $\mathcal{G}$ (V_0) be the one-dimensional space $\{\boldsymbol{y} : y_1 = \ldots = y_N\}$, and $\boldsymbol{G}$ be the orthogonal projection matrix onto $\mathcal{G}$. Then for any $\boldsymbol{y}$, $\boldsymbol{Gy}$ has all the enties equal to $\frac{1}{N}\sum_{i=1}^{N} y_i$, the overall mean.

$V_T \cap V_0^{\perp} = W_T$

W_B

W_C

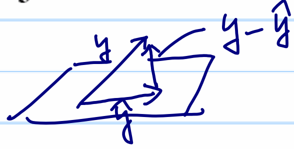
W_R

$$G = \frac{1}{N}\begin{bmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{bmatrix}$$

It is clear that $\mathcal{G} \subset \mathcal{F}$ for any factor F. Let $\mathcal{F} \ominus \mathcal{G}$ be the orthogonal complement of $\mathcal{G}$ relative to $\mathcal{F}$. Then dim $(\mathcal{F} \ominus \mathcal{G}) = f - 1$, and $\boldsymbol{F} - \boldsymbol{G}$ is the orthogonal projection matrix onto $\mathcal{F} \ominus \mathcal{G}$.

For instance, a typical entry of $(\boldsymbol{B} - \boldsymbol{G})\boldsymbol{y}$ is the deviation of a block average from the overall average.

The orthogonal projection matrix onto $\mathcal{F}^{\perp}$ is $\boldsymbol{I} - \boldsymbol{F}$.



Assume additivity between treatments and plots

$$Y_\omega = \alpha_{T(\omega)} + Z_\omega$$

$\alpha \leftrightarrow \tau$

Random-effect model (randomization model):

$$E(Y) = \alpha$$
$$cov(Y) = V$$

$\boldsymbol{V}$ has spectral form $\boldsymbol{V} = \sum_{i=0}^{s} \xi_i \boldsymbol{P}_i$

where $\boldsymbol{P}_i$ is the orthogonal projection matrix onto the eigenspace $\mathcal{S}_i$ of V with eigenvalue ξ_i depends on these values

The eigenspaces are independent of the values of the variances and covarinces: co-spectral.

$\sigma^2, \rho_1, \rho_2, \rho_3 \cdots$

Each of these eignespaces is called a stratum.

$$\mathcal{S}_0 = \mathcal{G}, \quad \boldsymbol{P}_0 = \boldsymbol{G} = \tfrac{1}{N}\boldsymbol{J}$$

$\mathcal{G} = V_0$

Completely randomized design

$$\boldsymbol{V} = a\boldsymbol{I} + b\boldsymbol{J} = a(\boldsymbol{I} - \tfrac{1}{N}\boldsymbol{J}) + \tfrac{1}{N}(a+Nb)\boldsymbol{J}$$
$$= a(\boldsymbol{I} - \boldsymbol{G}) + (a+Nb)\boldsymbol{G}.$$

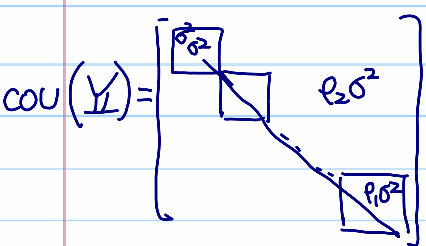
$$cov(\underline{Y}) = \begin{bmatrix} \sigma^2 & & \rho\sigma^2 \\ & \ddots & \\ \rho\sigma^2 & & \sigma^2 \end{bmatrix}$$

$$\therefore \ \boldsymbol{P}_0 = \boldsymbol{G}, \qquad \boldsymbol{P}_1 = \boldsymbol{I} - \boldsymbol{G},$$

$$\underset{V_0}{\mathcal{S}_0 = \mathcal{G}}, \qquad \underset{V_0^\perp}{\mathcal{S}_1 = \mathcal{G}^\perp}$$

Block design (b/k)

$$\boldsymbol{V} = \xi_2\boldsymbol{P}_2 + \xi_1\boldsymbol{P}_1 + \xi_0\boldsymbol{P}_0,$$



$$\underset{V_0}{\boldsymbol{P}_0 = \boldsymbol{G}}, \ \underset{W_B}{\boldsymbol{P}_1 = \boldsymbol{B} - \boldsymbol{G}}, \ \underset{V_B^\perp}{\boldsymbol{P}_2 = \boldsymbol{I} - \boldsymbol{B}},$$

$$\mathcal{S}_0 = \mathcal{G}, \mathcal{S}_1 = \mathcal{B} \ominus \mathcal{G}, \mathcal{S}_2 = \mathcal{B}^\perp.$$

$$\dim(\mathcal{S}_0) = 1, \ \dim(\mathcal{S}_1) = b-1, \ \dim(\mathcal{S}_2) = b(k-1)$$

Row-column design $(r \times c)$

$$\boldsymbol{V} = \xi_0\boldsymbol{P}_0 + \xi_1\boldsymbol{P}_1 + \xi_2\boldsymbol{P}_2 + \xi_3\boldsymbol{P}_3$$

$$\underset{V_0}{\boldsymbol{P}_0 = \boldsymbol{G}}, \ \underset{W_R}{\boldsymbol{P}_1 = \boldsymbol{R} - \boldsymbol{G}}, \ \underset{W_C}{\boldsymbol{P}_2 = \boldsymbol{C} - \boldsymbol{G}}, \ \underset{(V_R + V_C)^\perp}{\boldsymbol{P}_3 = (\boldsymbol{I} - \boldsymbol{R} - \boldsymbol{C} + \boldsymbol{G})}$$

$$\mathcal{S}_0 = \mathcal{G}, \quad \mathcal{S}_1 = \mathcal{R} \ominus \mathcal{G}, \quad \mathcal{S}_2 = \mathcal{C} \ominus \mathcal{G}, \quad \mathcal{S}_3 = (\mathcal{R} + \mathcal{C})^\perp$$

$$\dim(\mathcal{S}_0) = 1, \ \dim(\mathcal{S}_1) = r-1, \ \dim(\mathcal{S}_2) = c-1,$$
$$\dim(\mathcal{S}_3) = (r-1)(c-1)$$

$$R^N = \bigoplus_{i=0}^{s} S_i,\ \ S_i \perp S_j \text{ for all } i \neq j$$

$$\boldsymbol{I} = \sum_{i=0}^{s} \boldsymbol{P}_i,\ \ \boldsymbol{P}_i\boldsymbol{P}_j = \boldsymbol{0} \quad \text{for all } i \neq j$$

$$\boldsymbol{y} = \sum_{i=0}^{s} \boldsymbol{P}_i\boldsymbol{y} \qquad \boldsymbol{y} - \boldsymbol{G}\boldsymbol{y} = \sum_{i=1}^{s} \boldsymbol{P}_i\boldsymbol{y} \qquad \|\boldsymbol{y} - \boldsymbol{G}\boldsymbol{y}\|^2 = \sum_{i=1}^{s} \|\boldsymbol{P}_i\boldsymbol{y}\|^2$$

Total sum of squares

$$\boldsymbol{P}_0\boldsymbol{y} = \begin{bmatrix} \overline{y} \\ \vdots \\ \overline{y} \end{bmatrix} \qquad \|\boldsymbol{y} - \boldsymbol{P}_0\boldsymbol{y}\|^2 = \|\boldsymbol{y} - \boldsymbol{G}\boldsymbol{y}\|^2 = \sum_{i=1}^{N}(y_i - \overline{y})^2$$

Total variability

Null ANOVA → ignore treatment structure & only consider plot structure, i.e. only consider cov(Zw) E(Zw)

Suppose $\alpha_1 = \cdots = \alpha_t$ (in which case $\|\boldsymbol{y} - \boldsymbol{G}\boldsymbol{y}\|^2$ does not contain treatment effects, and therefore measures variability among the experimental units)

It can be shown that $$\mathrm{E}\left(\frac{1}{dim(S_i)}\|\boldsymbol{P}_i\boldsymbol{y}\|^2\right) = \xi_i$$

EMS

$E(P_i y) = 0$

ξ_i : ith stratum variance

Null ANOVA for a block design:

	Sources	SS	df	MS	E(MS)
W_B ←	interblock	$\sum_{i=1}^{b} k(y_{i\cdot} - y_{\cdot\cdot})^2$	$b-1$	$\frac{1}{b-1}\sum_{i=1}^{b} k(y_{i\cdot} - y_{\cdot\cdot})^2$	ξ_1 ← between-block variance
$W_B^\perp$ ←	intrablock	$\sum_{i=1}^{b}\sum_{j=1}^{k}(y_{ij} - y_{i\cdot})^2$	$b(k-1)$	$\frac{1}{b(k-1)}\sum_{i=1}^{b}\sum_{j=1}^{k}(y_{ij} - y_{i\cdot})^2$	ξ_2 ← within-block variance
	Total	$\sum_{i=1}^{b}\sum_{i=1}^{k}(y_{ij} - y_{\cdot\cdot})^2$	$bk-1$		

Null ANOVA table for a row-column design:

	Sources of variation	SS	df	MS	E(MS)
W_R ←	rows	$\sum_{i=1}^{r} c(y_{i\cdot} - y_{\cdot\cdot})^2$	$r-1$	$\frac{1}{r-1}\sum_{i=1}^{r} c(y_{i\cdot} - y_{\cdot\cdot})^2$	ξ_1 ← between row var.
W_C ←	columns	$\sum_{j=1}^{c} r(y_{\cdot j} - y_{\cdot\cdot})^2$	$c-1$	$\frac{1}{c-1}\sum_{j=1}^{c} r(y_{\cdot j} - y_{\cdot\cdot})^2$	ξ_2 ← between column var.
$(V_R + V_C)^\perp$ ←	units	$\sum_{i=1}^{r}\sum_{j=1}^{c}(y_{ij} - y_{i\cdot} - y_{\cdot j} + y_{\cdot\cdot})^2$	$(r-1)(c-1)$	$\frac{1}{(r-1)(c-1)}\sum_{i=1}^{r}\sum_{j=1}^{c}(y_{ij} - y_{i\cdot} - y_{\cdot j} + y_{\cdot\cdot})^2$	ξ_3 ←
	Total	$\sum_{i=1}^{r}\sum_{j=1}^{c}(y_{ij} - y_{\cdot\cdot})^2$	$rc-1$		

between-plot variance with row-to-row, column-to-column variance eliminated.

Nelder (1965a) gave simple rules for determining the degrees of freedom, projections onto the strata and the sums of squares in the null ANOVA for any simple orthogonal block structure.

Null ANOVA

nesting

N.

The first step is to determine how the total degrees of freedom are split up. For the block structure n_1/n_2, we have the following *d.f. identity*:

blocks   # of plots in a block

$$n_1 n_2 = 1 + \nu_1 + n_1 \nu_2,$$

where $\nu_i = n_i - 1$. The three terms on the right-hand side specify the degrees of freedom of the three strata.

crossing.

The d.f. identity for the block structure $n_1 \times n_2$ is

$$n_1 n_2 = 1 + \nu_1 + \nu_2 + \nu_1 \nu_2.$$

Define the *nesting* and *crossing* functions as

$$\mathcal{N}(n_1, n_2) = 1 + \nu_1 + n_1 \nu_2,$$

and

$$\mathcal{C}(n_1, n_2) = 1 + \nu_1 + \nu_2 + \nu_1 \nu_2,$$

$n_1/n_2/n_3$

$(n_1/n_2)/n_3$ $n_1/(n_2/n_3)$

$\mathcal{N}(\mathcal{N}(n_1,n_2),n_3)$ $\mathcal{N}(n_1,\mathcal{N}(n_2,n_3))$

The arguments in the $\mathcal{N}$ and $\mathcal{C}$ functions may be substituted by other $\mathcal{N}$ and $\mathcal{C}$ functions. The d.f. identities for other more complex block structures can be obtained from the block structure formulas by expanding the corresponding $\mathcal{N}$ and $\mathcal{C}$ functions. For example, the d.f. identity for $n_1/n_2/n_3$ can be obtained by expanding $\mathcal{N}(\mathcal{N}(n_1, n_2), n_3)$.

block

whole plot

sub plot

① Generally, terms cannot be destroyed by algebraic manipulation (e.g., $1 + \nu$ cannot be replace by n) except that ② like terms may be subtracted to become zero, which is deleted, and that ③ any unity appearing in a product is suppressed ($1 \cdot x = x$).

④ Another important rule: *the n_1 term that appears in the nesting function formula must be the algebraic sum of all the terms in the expansion of n_1.*

$n_1/n_2/n_3$ $\mathcal{N}(\mathcal{N}(n_1, n_2), n_3) = 1 + (\mathcal{N}(n_1, n_2) - 1) + n_1 n_2 \nu_3$

$1 + (\mathcal{N}(n_1,n_2) - 1) + n_1 \cdot n_2 \cdot \nu_3$

$1 + \nu_1 + n_1 \cdot \nu_2$ $\because (4)$

$= 1 + \nu_1 + n_1 \cdot \nu_2 + n_1 n_2 \cdot \nu_3$

$$= 1 + \underset{n_1 - 1}{\nu_1} + \underset{n_1(n_2-1)}{n_1 \nu_2} + \underset{n_1 n_2 (n_3 - 1)}{n_1 n_2 \nu_3}.$$

So there are four strata with degrees of freedom 1, $n_1 - 1$, $n_1(n_2 - 1)$ and $n_1 n_2 (n_3 - 1)$.

From the d.f. identity, we can write down a yield identity which gives projections to all the strata. For convenience, we index each plot by multi-subscripts, and as before, dot notation is used for averaging. The following is the rule given by Nelder:

Expand each term in the d.f. identity as a function of the n's; then to each term in the expansion corresponds a mean of the y's with the same sign and averaged over the subscripts for which the corresponding n's are absent.

$(n_1/n_2)/n_3$:

$$n_1 n_2 n_3 = 1 + (n_1 - 1) + n_1(n_2 - 1) + n_1 n_2 (n_3 - 1)$$
$$= 1 + (n_1 - 1) + (n_1 n_2 - n_1) + (n_1 n_2 n_3 - n_1 n_2) \leftarrow \text{d.f. identity.}$$

This gives the following yield identity:

$$\sum_{i,j,k} y_{ijk} = \sum_{ijk} y_{...} + \sum_{ijk} (y_{i..} - y_{...}) + (y_{ij.} - y_{i..}) + (y_{ijk} - y_{ij.})$$

from block factor.

$\therefore$ The strata other than $\mathcal{G}$ have degrees of freedom equal to $n_1 - 1$, $n_1 n_2 - n_1$ and $n_1 n_2 n_3 - n_1 n_2$. The corresponding sums of squares in the null ANOVA are $\sum_{i=1}^{n_1} n_2 n_3 (y_{i..} - y_{...})^2$, $\sum_{i=1}^{n_1}\sum_{j=1}^{n_2} n_3 (y_{ij.} - y_{i..})^2$, and $\sum_{i=1}^{n_1}\sum_{j=1}^{n_2}\sum_{k=1}^{n_3} (y_{ijk} - y_{ij.})^2$.

$\|P_i y\|^2$ = sum of square

Null ANOVA for $n_1/n_2/n_3$

$$\sum_{i=1}^{n_1}\sum_{j=1}^{n_2}\sum_{k=1}^{n_3}(y_{ijk}-y_{...})^2=\sum_{i=1}^{n_1}n_2n_3(y_{i..}-y_{...})^2+\sum_{i=1}^{n_1}\sum_{j=1}^{n_2}n_3(y_{ij.}-y_{i..})^2 + \sum_{i=1}^{n_1}\sum_{j=1}^{n_2}\sum_{k=1}^{n_3}(y_{ijk}-y_{ij.})^2$$

If $\alpha_1=\cdots=\alpha_t$, then

$$\mathrm{E}[\frac{1}{n_1-1}\sum_{i=1}^{n_1}n_2n_3(y_{i..}-y_{...})^2]=\xi_1 : \text{between-block variance}$$

$$\mathrm{E}[\frac{1}{n_1(n_2-1)}\sum_{i=1}^{n_1}\sum_{j=1}^{n_2}n_3(y_{ij.}-y_{i..})^2]=\xi_2 : \text{between-wholeplot variance}$$

$$\mathrm{E}[\frac{1}{n_1n_2(n_3-1)}\sum_{i=1}^{n_1}\sum_{j=1}^{n_2}\sum_{k=1}^{n_3}(y_{ijk}-y_{ij.})^2]=\xi_3 : \text{within-wholeplot variance}$$

block factors

$1+\nu_1+n_1(\underbrace{C(n_2,n_3)}_{x+\nu_2+\nu_3+\nu_2\nu_3}-x)$

$n_1/(n_2\times n_3)$:

$$\mathcal{N}(n_1,\mathcal{C}(n_2,n_3))=1+\nu_1+n_1(1+\nu_2+\nu_3+\nu_2\nu_3-1)$$
$$=1+\nu_1+n_1\nu_2+n_1\nu_3+n_1\nu_2\nu_3$$

d.f. identity:

$$n_1n_2n_3=1+(n_1-1)+n_1(n_2-1)+n_1(n_3-1)+n_1(n_2-1)(n_3-1)=$$

$$1+(n_1-1)+(n_1n_2-n_1)+(n_1n_3-n_1)+(n_1n_2n_3-n_1n_2-n_1n_3+n_1)$$

yield identity:

$$y_{ijk}=y_{...}+(y_{i..}-y_{...})+(y_{ij.}-y_{i..})+(y_{i.k}-y_{i..}) + (y_{ijk}-y_{ij.}-y_{i.k}+y_{i..})$$

Miller (1997) *Technometrics*

$\therefore$ The strata other than $\mathcal{G}$ have degrees of freedom equal to $n_1-1, n_1n_2-n_1, n_1n_3-n_1$ and $n_1n_2n_3-n_1n_2-n_1n_3+n_1$. The corresponding sums of squares in the null ANOVA are

$$\sum_{i=1}^{n_1} n_2n_3(y_{i\cdot\cdot}-y_{\cdot\cdot\cdot})^2,\ \sum_{i=1}^{n_1}\sum_{j=1}^{n_2} n_3(y_{ij\cdot}-y_{i\cdot\cdot})^2,\ \sum_{i=1}^{n_1}\sum_{k=1}^{n_3} n_2(y_{i\cdot k}-y_{i\cdot\cdot})^2,$$

and $\sum_{i=1}^{n_1}\sum_{j=1}^{n_2}\sum_{k=1}^{n_3}(y_{ijk}-y_{ij\cdot}-y_{i\cdot k}+y_{i\cdot\cdot})^2$.

$n_1/((n_2/n_3/n_4)\times n_5)$

$$\mathcal{N}(n_1,\mathcal{C}(\mathcal{N}(\mathcal{N}(n_2,n_3),n_4),n_5))$$

$$= 1+\nu_1+n_1(\underline{\mathcal{C}(\mathcal{N}(\mathcal{N}(n_2,n_3),n_4),n_5)}\ \cancel{-1})$$

$$= \cancel{1} + (N(N(n_2,n_3),n_4)-1) + \nu_5 + (N(N(n_2,n_3),n_4)-1)\cdot\nu_5$$

$$= 1+\nu_1+n_1((\mathcal{N}(\mathcal{N}(n_2,n_3),n_4)-1)+\nu_5+(\underline{\mathcal{N}(\mathcal{N}(n_2,n_3),n_4)}\ \cancel{-1})\nu_5)$$

$$= \cancel{1} + (N(n_2,n_3) - \cancel{1}) + n_2n_3\nu_4$$

$$= 1+\nu_1+n_1(\nu_2+n_2\nu_3+n_2n_3\nu_4+\nu_5+(\nu_2+n_2\nu_3+n_2n_3\nu_4)\nu_5)$$

$$= 1+\nu_1+n_1\nu_2+n_1n_2\nu_3+n_1n_2n_3\nu_4+n_1\nu_5+n_1\nu_2\nu_5+n_1n_2\nu_3\nu_5 + n_1n_2n_3\nu_4\nu_5$$

$$\cancel{1} + \nu_2 + n_2\nu_3$$

There are 8 nontrivial strata

Null ANOVA table $\longrightarrow$ only plot structure

i.e. decomposition based on $\mathrm{cov}(\underline{Y})$

Sources of variation	SS	df	MS	E(MS)
$\mathcal{S}_1$	$\lVert P_1 y\rVert^2$	$\dim(\mathcal{S}_1)$	$\frac{1}{dim(\mathcal{S}_1)}\lVert P_1 y\rVert^2$	ξ_1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\mathcal{S}_s$	$\lVert P_s y\rVert^2$	$\dim(\mathcal{S}_s)$	$\frac{1}{dim(\mathcal{S}_s)}\lVert P_s y\rVert^2$	ξ_s
Total	$\lVert y-Gy\rVert^2$	$N-1$		

$\longleftrightarrow \sigma^2, \rho_1, \rho_2, \rho_3$

W_T

Designs such that $\mathcal{T}\ominus\mathcal{G}$ falls entirely in one stratum are called orthogonal designs.

Examples: Completely randomized designs
Randomized complete block designs
Latin squares, row-column design

We say that F_1 and F_2 satisfy the condition of *proportional frequencies* if

treatment factor

block factor

$$n_{ij} = \frac{n_{i+}n_{+j}}{n_{++}} \quad \text{for all } i, j,$$

where $n_{i+} = \sum_{j=1}^{f_2} n_{ij}$, $n_{+j} = \sum_{i=1}^{f_1} n_{ij}$ and $n_{++} = \sum_{i=1}^{f_1}\sum_{j=1}^{f_2} n_{ij} = N$.

Theorem. Two factors F_1 and F_2 satisfy the condition of proportional frequencies $\Leftrightarrow \mathcal{F}_1 \ominus \mathcal{G} \perp \mathcal{F}_2 \ominus \mathcal{G}$.

Under a row-column design such that each treatment appears the same number of times in each row and the same number of times in each each column (such as a Latin square),

$$(\mathcal{T} \ominus \mathcal{G}) \perp (\mathcal{R} \ominus \mathcal{G})$$
$$(\mathcal{T} \ominus \mathcal{G}) \perp (\mathcal{C} \ominus \mathcal{G})$$

$$\mathcal{T} \ominus \mathcal{G} \subset (\mathcal{R} + \mathcal{C})^{\perp} (= \mathcal{S}_3)$$

Under alternative, $\alpha_1, \cdots, \alpha_t$ not all equal

If $\mathcal{T} \ominus \mathcal{G} \subset \mathcal{S}_i$ for some i, then

$$\|\boldsymbol{P}_i \boldsymbol{y}\|^2 = \|(\boldsymbol{T} - \boldsymbol{G})\boldsymbol{y}\|^2 + \|P_{\mathcal{S}_i \ominus (\mathcal{T} \ominus \mathcal{G})} \boldsymbol{y}\|^2$$

$$= \underbrace{\sum_{j=1}^{t} r_j(\overline{T}_j - y_{\cdot})^2}_{\text{Variation of treatment mean}} + \|P_{\mathcal{S}_i \ominus (\mathcal{T} \ominus \mathcal{G})} \boldsymbol{y}\|^2,$$

where r_j is the number of replications of the jth treatment and $\overline{T}_j$ is the jth treatment mean.

treatment sum of squares + residual sum of squares

$E(\|P_{W_T} y\|^2)/\dim(W_T) = \dfrac{\|\tau_T\|^2}{\dim(W_T)}$, $\dim(W_T) = t-1$

$$E[\tfrac{1}{t-1}\sum_{j=1}^{t} r_j(\overline{T}_j - y_{\cdot})^2] = \xi_i + \boxed{\tfrac{1}{t-1}\sum_{j=1}^{t} r_j[\alpha_j - \tfrac{1}{N}\sum_{j=1}^{t} r_j\alpha_j]^2}$$

$$E[\tfrac{1}{\dim(\mathcal{S}_i)-t+1}\|P_{\mathcal{S}_i \ominus (\mathcal{T} \ominus \mathcal{G})} \boldsymbol{y}\|^2] = \xi_i$$

Sources	SS	df	MS	E(MS)
$\mathcal{S}_1$	$\lVert \boldsymbol{P}_1\boldsymbol{y}\rVert^2$	$\dim(\mathcal{S}_1)$	$\frac{1}{dim(\mathcal{S}_1)}\lVert \boldsymbol{P}_1\boldsymbol{y}\rVert^2$	ξ_1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\mathcal{S}_i$				
Treatments	$\cdots$	$t-1$	$\cdots$	$\xi_i+\cdots$
Residual	$\cdots$	$\dim(\mathcal{S}_i)-t+1$	$\cdots$	ξ_i
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\mathcal{S}_s$	$\lVert \boldsymbol{P}_s\boldsymbol{y}\rVert^2$	$\dim(\mathcal{S}_s)$	$\frac{1}{dim(\mathcal{S}_s)}\lVert \boldsymbol{P}_s\boldsymbol{y}\rVert^2$	ξ_s
Total	$\lVert \boldsymbol{y}-\boldsymbol{G}\boldsymbol{y}\rVert^2$	$N-1$		

ANOVA table for a Latin square design:

Sources of variation	SS	d.f.	MS	E(MS)
Rows	$\sum_{i=1}^{t} t(y_{i.}-y_{..})^2$	$t-1$	$\cdots$	ξ_1
Columns	$\sum_{j=1}^{t} t(y_{.j}-y_{..})^2$	$t-1$	$\cdots$	ξ_2
Treatments	$\sum_{i=1}^{t} t[\bar{T}_i-y_{..}]^2$	$t-1$	$\cdots$	$\xi_3+\frac{1}{t-1}[\sum_{i=1}^{t} t(\alpha_i-\alpha_.)^2]$
Residual	By subtraction			ξ_3
Total	$\sum_{i=1}^{t}\sum_{j=1}^{t}(y_{ij}-y_{..})^2$	t^2-1		

3 strata