(Bailey) Sec. 4.4 models for block designs

(Q: how to model Z_ω: the effect of plot structure)

- **Fixed effect model**

$$Z_\omega = \underbrace{\zeta_{B(\omega)}}_{\substack{\text{fixed parameters} \\ \text{(c.f. treatment effect)}}} + \underbrace{e_\omega}_{\substack{\text{random variable} \\ E(e_\omega)=0 \\ cov(e_\omega, e_{\omega'})=0}}$$

Q: when to use fixed effect

- blocks = population
- conditional conclusion on the blocks
- block difference not change, say over time.

Then, $E(Z_\omega) = \zeta_{B(\omega)}$

$$cov(Z_\omega, Z_{\omega'}) = \begin{cases} 0 & \text{if } \omega \neq \omega' \\ \sigma^2, & \text{if } \omega = \omega' \end{cases}$$

$\boxed{Y = \tau_{T(\omega)} + Z_\omega}$

$$\Rightarrow \begin{cases} E(Y) = \tau + \zeta \\ cov(Y) = \sigma^2 I \end{cases}$$

- **random effect model**

$$Z_\omega = \underbrace{\zeta_{B(\omega)}}_{\substack{\text{random} \\ \text{variable}}} + \underbrace{e_\omega}_{\substack{\text{random} \\ \text{variable}}}$$

$$= \zeta_0 + (\zeta_{B(\omega)} - \zeta_0) + e_\omega$$

- ζ_0, $\zeta_{B(\omega)} - \zeta_0$, e_ω: uncorrelated, mean zero, variance $\sigma_0^2, \sigma_B^2, \sigma_e^2$, respectively.

Q: when to use random effect?

- blocks $\ll$ population
- conclusion would like to be applied on the population
- block difference change, say over time ...

Q: do we need to restrict sum of $\zeta_{B(\omega)} - \zeta_0 = 0$?
(If ζ_0: average of $\zeta_{B(\omega)}$, yes; If ζ_0: average of whole population, no)

$E(Z_\omega) = 0$

$$cov(Z_\alpha, Z_\beta) = \begin{cases} \underbrace{\sigma_0^2 + \sigma_B^2 + \sigma_e^2}_{=\sigma^2} & \alpha = \beta \\ \underbrace{\sigma_0^2 + \sigma_B^2}_{=\rho_1\sigma^2} & \alpha \neq \beta,\ B(\alpha) = B(\beta) \\ \underbrace{\sigma_0^2}_{=\rho_2\sigma^2} & B(\alpha) \neq B(\beta) \end{cases}$$

$$\Rightarrow \begin{bmatrix} \begin{bmatrix} \sigma^2 & \rho_1\sigma^2 \\ \rho_1\sigma^2 & \sigma^2 \end{bmatrix} & & & \rho_2\sigma^2 \\ & \begin{bmatrix} \sigma^2 & \rho_1\sigma^2 \\ \rho_1\sigma^2 & \sigma^2 \end{bmatrix} & & \\ & & \ddots & \\ \rho_2\sigma^2 & & & \square \end{bmatrix}$$

$\Rightarrow \quad E(\mathbf{Y}) = \mathbb{T}$

$\quad cov(\mathbf{Y})$ = the matrix in previous slide

- Let J_B be the $\underset{\text{\# of plots}}{N}\times N$ matrix whose (α,β) entry is $\begin{cases} 1, & \text{if } B(\alpha)=B(\beta) \\ 0, & \text{o.w.} \end{cases}$

$$J_B = \begin{bmatrix} \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{bmatrix} & & & 0 \\ & \boxed{\mathbb{1}} & & \\ & & \ddots & \\ 0 & & & \boxed{\mathbb{1}} \end{bmatrix}$$

Then $cov(\mathbf{Y}) = \sigma^2 I + \rho_1\sigma^2 (J_B - I) + \rho_2\sigma^2 (\underset{\begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{bmatrix}_{N\times N}}{J} - J_B)$

$\qquad = \sigma^2\left[(1-\rho_1)I + (\rho_1-\rho_2)J_B + \rho_2 J\right]$

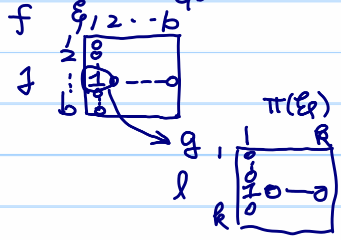
- An alternative approach: ζ_0 regarded as fixed effect.

$\qquad \Rightarrow \rho_2 = 0$ & ζ_0 goes into $E(\mathbf{Y})$

(Reading material 02, Sec 1.3) Randomization model

- randomness comes from the restricted randomization procedure in plan.
- b blocks, labeled by $\xi_q = 1, 2, \cdots, b$
- k plots in block ξ_q, labeled by $\pi(\xi_q) = 1, 2, \cdots, k$
- randomization of blocks:
  - choose a permutation of $\{1, \cdots, b\}$
  - after the permutation, relabel blocks by $j = 1, 2, \cdots, b$
- randomization of plots in block ξ_q
  - (for each block) choose a permutation of $\{1, 2, \cdots, k\}$
  - after permutation, relabel the plots by $\ell = 1, 2, \cdots, k$
- Let $\underbrace{f_{j\xi_q}}_{\text{random variable}} = \begin{cases} 1, & \text{if block originally labeled by } \xi_q \text{ is relabel by } j \\ 0, & \text{o.w.} \end{cases}$

$\quad \underbrace{g_{\ell\pi(\xi_q)}}_{\text{random variable}} = \begin{cases} 1, & \text{if plot } \vdots \quad \vdots \quad \vdots\ \pi(\xi_q)\ \vdots \quad \vdots \quad \vdots\ \ell \\ 0, & \end{cases}$



(*1) $\sum_{\xi_q=1}^{b} f_{j\xi_q} = 1, \quad \sum_{\pi(\xi_q)=1}^{k} g_{\ell\pi(\xi_q)} = 1$

$\qquad \sum_{j=1}^{b} f_{j\xi_q} = 1, \quad \sum_{\ell=1}^{k} g_{\ell\pi(\xi_q)} = 1$

(*2) $E(f_{j\xi_1}) = 1/b$ $\quad E(f_{j\xi_1} f_{j'\xi_1'}) = \begin{cases} 1/b & , \ j=j', \xi_1=\xi_1' \\ 0 & \ j=j', \xi_1\neq\xi_1' \\ 0 & \ j\neq j', \xi_1=\xi_1' \\ 1/b(b-1) & \ j\neq j', \xi_1\neq\xi_1' \end{cases}$

$E(g_{\ell\pi(\xi_1)}) = \frac{1}{k}$ $\quad E(g_{\ell\pi(\xi_1)} g_{\ell'\pi'(\xi_1)}) = \begin{cases} 1/k \\ 0 \\ 0 \\ 1/k(k-1) \end{cases}$

because of the independence of $\overset{\text{plot}}{\wedge}$ randomization in different blocks

$\therefore E(g_{\ell\pi(\xi_1)} g_{\ell'\pi(\xi_1')}) = E(\quad)E(\quad) = \frac{1}{k^2}$, $\xi_1 \neq \xi_1'$

(*3) $\because$ randomization of blocks is independent of the plot randomization within block

$E(f_{j\xi_1} g_{\ell\pi(\xi_1')}) = E(\quad)E(\quad) = \frac{1}{b}\cdot\frac{1}{k}$

for $\xi_1 = \xi_1'$ or $\xi_1 \neq \xi_1'$

- Let the response contributed by the plot originally labeled by $\pi(\xi_1)$ be denoted by $\mu_{\pi(\xi_1)}$.

Let the block of the plot receives label j after block randomization and the plot receives label ℓ after plot randomization with block.

$$m_{\ell(j)} = \sum_{\xi_1=1}^{b} \sum_{\pi(\xi_1)=1}^{k} f_{j\xi_1} g_{\ell\pi(\xi_1)} \cdot \mu_{\pi(\xi_1)}$$

- Introduce the identity:

$$\mu_{\pi(\xi_1)} = \mu_{\cdot(\cdot)} + [\mu_{\cdot(\xi_1)} - \mu_{\cdot(\cdot)}] + [\mu_{\pi(\xi_1)} - \mu_{\cdot(\xi_1)}]$$

where $\mu_{\cdot(\xi_1)} = \frac{1}{k}\sum_{\pi(\xi_1)=1}^{k} \mu_{\pi(\xi_1)}$ ← average of plot contribution within block ξ_1

$\mu_{\cdot(\cdot)} = \frac{1}{b}\sum_{\xi_1=1}^{b} \mu_{\cdot(\xi_1)}$ ← average of all plots contribution (bk plots)

- Then,

$$m_{\ell(j)} = \underbrace{\sum_{\xi_1=1}^{b}\sum_{\pi(\xi_1)=1}^{k} f_{j\xi_1} g_{\ell\pi(\xi_1)} \mu_{\cdot(\cdot)}}_{\mu_{\cdot(\cdot)} \equiv \mu} + \underbrace{\sum\sum f_{j\xi_1} g_{\ell\pi(\xi_1)} (\mu_{\cdot(\xi_1)} - \mu_{\cdot(\cdot)})}_{= \sum_{\xi_1=1}^{b} f_{j\xi_1}(\mu_{\cdot(\xi_1)} - \mu_{\cdot(\cdot)}) \equiv \beta_j} + \underbrace{\sum\sum f_{j\xi_1} g_{\ell\pi(\xi_1)} (\mu_{\pi(\xi_1)} - \mu_{\cdot(\xi_1)})}_{\equiv \tau_{\ell(j)}}$$

Zw

$\Rightarrow m_{\ell(j)} = \mu + \beta_j + \tau_{\ell(j)}$

where μ is a constant parameter and

β_j : random variable ⇒ block random effect

↳ because of the randomization of blocks

Note that $\sum_{j=1}^{b} \beta_j = 0$ (⇒ sum of random effects = 0)

$\eta_{\ell(j)}$: unit (plot) error : random variable

because of the block randomization & plot randomization

• It is easy to get (from (*1), (*2), (*3))

(1) $E(\beta_j)=0$ & $E(\eta_{\ell(j)})=0$

(2) $cov(\beta_j, \beta_{j'}) = \begin{cases} \frac{1}{b} \sum_{\xi_1=1}^{b} (\mu_{\cdot(\xi_1)} - \mu_{\cdot(\cdot)})^2, & j=j' \\ -\frac{1}{b(b-1)} \boxed{\sum(\quad)^2} \equiv \sigma_B^2, & j \neq j' \end{cases}$

c.f. random effect model & the restriction $\sum \beta_j = 0$

(3) $cov(\eta_{\ell(j)}, \eta_{\ell'(j')}) = \begin{cases} \frac{1}{bk} \sum_{\xi_1=1}^{b} \sum_{\pi(\xi_1)=1}^{k} (\mu_{\pi(\xi_1)} - \mu_{\cdot(\xi_1)})^2, & j=j', \ell=\ell' \\ 0 \\ 0 \\ -\frac{1}{k} \cdot \frac{1}{b(k-1)} \boxed{\sum\sum(\quad)^2} \equiv \sigma_U^2, & j=j', \ell \neq \ell' \end{cases}$

(4) $cov(\beta_j, \eta_{\ell(j')}) = 0$

• From (1), (2), (3), (4), we can get

$E(m_{\ell(j)}) = \mu$

c.f. the matrix in LNp.3-13

$cov(m_{\ell(j)}, m_{\ell'(j')}) = \begin{cases} \sigma^2, & \text{if } j=j', \ell=\ell' \\ \rho_1 \sigma^2, & j=j', \ell \neq \ell' \\ \rho_2 \sigma^2, & j \neq j' \end{cases} = (\delta_{jj'} - \frac{1}{b})\sigma_B^2 + (\delta_{jj'}(\delta_{\ell\ell'} - \frac{1}{k}))\sigma_U^2$

$\sigma^2 = \frac{b-1}{b}\sigma_B^2 + \frac{k-1}{k}\sigma_U^2$

$\rho_1 = \left[\frac{b-1}{b}\sigma_B^2 - \frac{1}{k}\sigma_U^2\right] / \sigma^2$

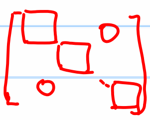
$\rho_2 = -\frac{1}{b}\sigma_B^2 / \sigma^2$

⇒ dimension = 2

$\begin{bmatrix} \sigma^2 & & 0 \\ & \ddots & \\ 0 & & \sigma^2 \end{bmatrix}$ ← • fixed effect • random effect

not zero because of $\sum \beta_j = 0$

c.f. the last point in LNp3-14



*sum of block effect ≠ 0

← fixed intercept effect

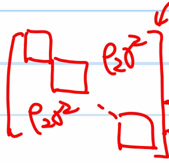
random intercept effect

• add a technical error $e_{\ell(j)}$

$Z_{\ell(j)} = m_{\ell(j)} + e_{\ell(j)}$ ↳ $var = \sigma_e^2$

$E(Z_{\ell(j)}) = \mu$

Z_w



*Sum of block effect = 0

← fixed --

← random

dimension=3 ← $cov(Z_{\ell(j)}) = (\delta_{jj'} - \frac{1}{b})\sigma_B^2 + [\delta_{jj'}(\delta_{\ell\ell'} - \frac{1}{k})] \cdot \sigma_U^2 + \delta_{jj'}\delta_{\ell\ell'}\sigma_e^2$

jointly made by Ching-Shui Cheng (Berkeley) and Shao-Wei Cheng (NTHU)