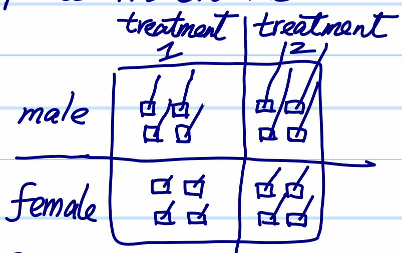
Blocking: an effective method for improving precision

avoid

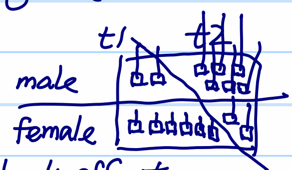
- heterogeneous
  If plots are not all reasonable similar, we should group them into blocks in such a way that
  plots within each block are alike
  homogeneous within block
- Block what you can and randomize what you cannot.
- The purpose of randomization is to average out those nuisance factors that we cannot predict or cannot control, not to destroy the relevant information we have

Q: What's the difference between treatment factor and block factor?
↳ interested in its effect.

Q: Why do we need to arrange plots in blocks?



* if blocks are not observed, variation caused by blocks increase the variance estimation of ε (error term)



* block effect causes bias on the estimation of treatment effect (mean structure)

Q: how to block? If possible

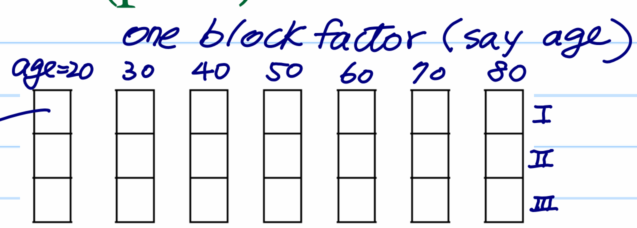
(i) block should have the same size (# of plots in a block)

(ii) block size should be big enough to allow all treatments to occur at least once in each block.
(⇒ allow within-block comparison)

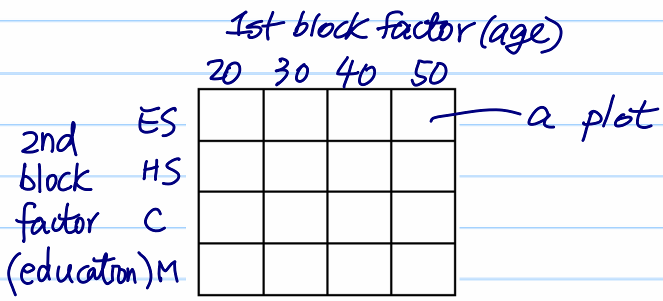
Reading Assignment: Bailey. Sec. 4.1

Two simple block (plot) structures

- Nesting
  block/plot



- Crossing
  row ∗ column



Simple orthogonal block structures

- Iterated crossing and nesting
- cover most, but not all block structures encountered in practice

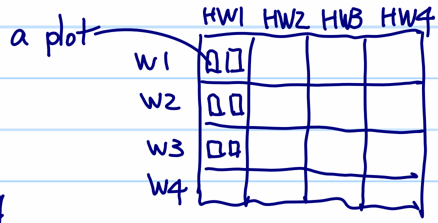
Example1: Consumer testing

A consumer organization wishes to compare 8 brands of vacuum cleaner. There is one sample for each brand. Each of four housewives tests two cleaners in her home for a week. To allow for housewife effects, each housewife tests each cleaner and therefore takes part in the trial for 4 weeks.

→ block factor (housewives)

weeks. → block factor

Q: What is the plot?



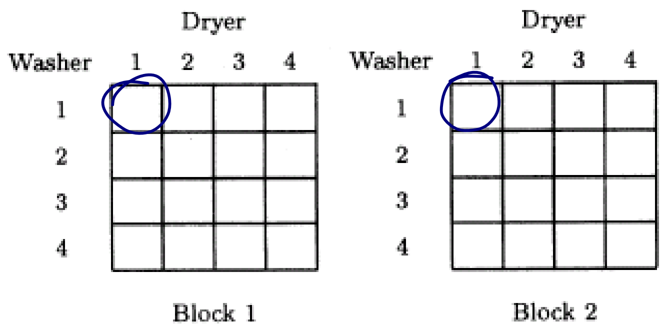
Block structure: crossing, nesting

$$(4 \times 4)/2$$

Example 2: Miller (1997, Technometrics)

The experiment is run in 2 blocks and employs 4 washers and 4 driers. Sets of cloth samples are run through the washers and the samples are divided into groups such that each group contains exactly one sample from each washer. Each group of samples is then assigned to one of the driers. Once dried, the extent of wrinkling on each sample is evaluated.

Q: what is the plot?

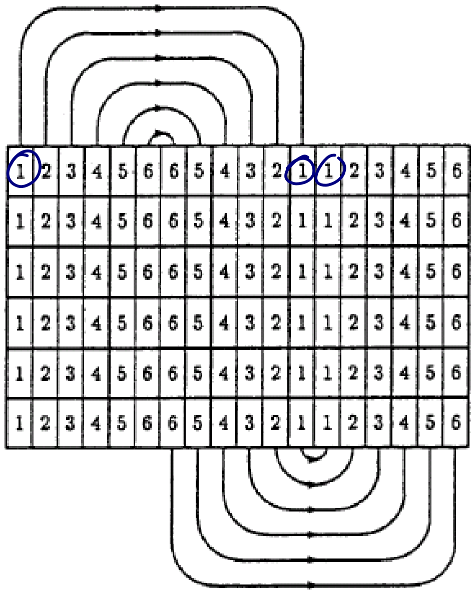


Block structure:

Q: Can we say it's (2 blocks * 4 washers * 4 dryers)?

$$2 \text{ blocks}/(4 \text{ washers} * 4 \text{ dryers})$$

a strip



block factor
- strip
- all 1's
  all 2's
  ⋮
  all 6's

strip 1 ... 6
label 1
label 6

Block structure:

$$(6 \times 6)/3$$

3 treatments:

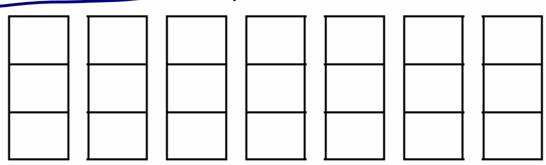
1	2	3	1	2	3	1	2	3

After randomization:

1	3	2	2	1	3	2	3	1

Randomized complete block design
↳ block size ≥ # of treatments

Incomplete block design ≈ < ≈



7 treatments

1	2	3	4	5	6	7
2	3	4	5	6	7	1
4	5	6	7	1	2	3

For the rest of lecture note 3, we focus on *nesting* and *complete block* plot structure

(Bailey) Sec. 4.2. Orthogonal Block Designs.

- We now suppose that Ω consists of b blocks of equal size k
- define a block factor B on Ω by
  $B(\omega)$ = the block containing ω.
- The block space V_B (c.f. V_T) consist of those vectors in V which takes a constant value on each block
  - For $j=1,\cdots,b$, let v_j be the vector whose entry on plot ω is equal to

$$v_j = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \text{ } j\text{th block} \qquad \begin{cases} 1, & \text{if } \omega \text{ is in block } j, \text{ i.e. } B(\omega)=j \\ 0, & \text{o.w.} \end{cases}$$

(c.f. u_i in lecture notes 02)

- Then, $v_i \cdot v_j = \begin{cases} k, & \text{if } i=j \\ 0, & \text{if } i\neq j \end{cases}$
- $\{v_1,\cdots,v_b\}$ is an orthogonal basis for V_B
  - $\dim(V_B)=b$

- $u_0 = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = \sum_{i=1}^{t} u_i = \sum_{j=1}^{b} v_j \Rightarrow V_0 \subset V_B$
- define $W_B = \{v \in V_B \mid v \perp u_0\} = V_B \cap V_0^\perp$ (c.f. $W_T = V_T \cap V_0^\perp$)

Def: A block design is called orthogonal if the subspaces W_T and W_B are orthogonal to each other.

Theorem 4.1 *Let s_{ij} be the number of times that treatment i occurs in block j, for $i = 1, \ldots, t$ and $j = 1, \ldots, b$. Then the block design is orthogonal if and only if $s_{ij} = r_i/b$ for $i = 1, \ldots, t$ and $j = 1, \ldots, b$.*

$r_i \to$ # of replicates of treatment i

$s_{ij} \to$ a constant over j

proof:
- $s_{ij} = u_i \cdot v_j$
- $\because W_T \perp V_0 \quad \therefore W_T \perp W_B \Leftrightarrow W_T \perp (V_B) = W_B \oplus V_0$

(orthogonal block design) $\Leftrightarrow \underbrace{\left(\sum_{i=1}^{t} a_i u_i\right) \cdot v_j}_{= \sum_{i=1}^{t} a_i s_{ij}} = 0$

$W_T \ni \left(\sum_{i=1}^{t} a_i u_i\right)$, $\left(\sum_{i=1}^{t} a_i u_i\right) \cdot u_0 = 0 = \sum_{i=1}^{t} a_i r_i$

$\Leftrightarrow \sum_{i=1}^{t} a_i s_{ij} = 0$ whenever $\sum_{i=1}^{t} a_i r_i = 0 \overset{?}{\Leftrightarrow} s_{ij} = \frac{r_i}{b}$

(⇐) If $s_{ij} = r_i/b$, then $\sum a_i s_{ij} = \sum a_i r_i/b = 0$

(⇒) Suppose that $W_T \perp W_B$. Fix an i, where $i \neq 1$,

Let $a_1^* = 1/r_1$, $a_i^* = -1/r_i$, $a_l^* = 0$ for $l \notin \{1, i\}$

Then (1) $\sum a_i^* r_i = r_1 \cdot (1/r_1) + r_i(-1/r_i) = 0$

(2) $0 = \sum a_i^* s_{ij} = s_{1j}/r_1 - s_{ij}/r_i$

$\Rightarrow s_{ij} = r_i s_{1j}/r_1$ for all i and all j

Counting the # of plots in a block j

$$k = \sum_{i=1}^{t} s_{ij} = \sum_{i=1}^{t} r_i s_{1j}/r_1 = \frac{s_{1j}}{r_1}\underbrace{\left(\sum_{i=1}^{t} r_i\right)}_{bk = |\Omega| = N} = \frac{s_{1j}}{r_1} bk$$

$\Rightarrow s_{1j} = r_1/b$

$\Rightarrow$ hence, $s_{ij} = r_i/b$ for all i.

Def: A complete-block design has blocks of size t ($\to m \cdot t$), with each treatment occurring once (m times) in each block.

i.e., $s_{ij} = 1$ $\forall i \,\&\, j$ and $r_i = b$.

Corollary 4.2: complete-block design is orthogonal.

(Bailey)

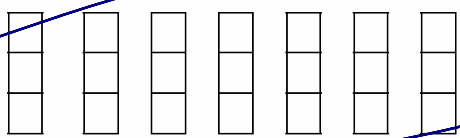
Sec. 4.3. Construction and Randomization

(i) Apply treatment i to r_i/b plots in block 1, for $i = 1, \ldots, t$, and randomize, just as for a completely randomized design.

→ randomize the order of block

(ii) Repeat for each block, using a fresh randomization each time, independent of the preceding randomizations.

- An example:

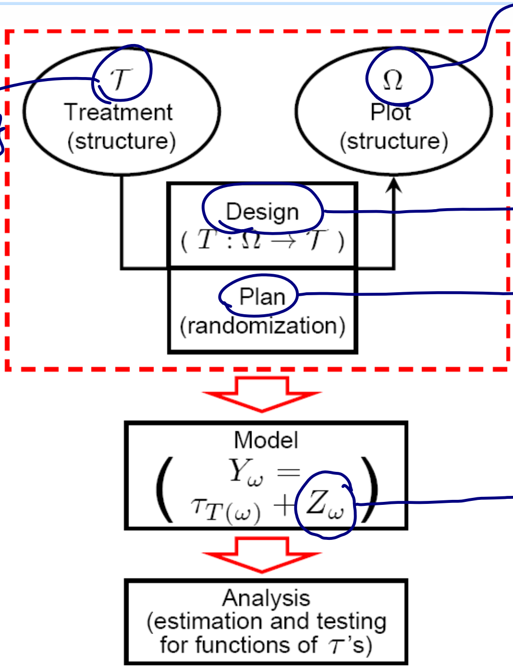


→ restricted randomization becaus of block structure.

Randomize by randomly choosing one out of the $(7!)(3!)^7$ permutations that preserve the block structure.
(Note: These permutations form a subgroup of the group of all 21! permutations of the 21 unit labels.)

larger

→ unrestricted randomization.



$\{1, 2, \cdots, t\}$

nesting block structure

b blocks/k plots

$k = t, 2t, 3t, \cdots$

complete-block design

restricted randomization as described in LN3-10

- fixed-effect model
- random-effect model
- randomization model

block effect

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