(Reading Material 02) Sec. 1.2. A Randomization model for Completely Randomized Design (c.f. the linear model in LNp.2-10)

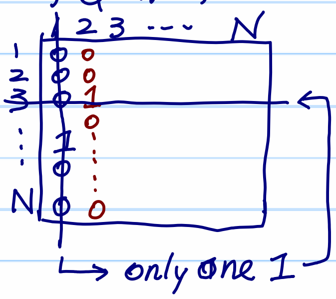
- N plots: labels by $\xi = 1, 2, \cdots, N$
- Randomization: choose a random permutation of $\{1, 2, \cdots, N\}$ (LNp.2-1)

$\xi =$ 1 2 ... N
$j =$ 10 8 5

- Once a permutation is selected, the plots are renumbered with $j = 1, \cdots, N$
- Let $f_{j\xi} = \begin{cases} 1, & \text{if plot originally labelled by } \xi \text{ receives by the randomization the label } j \\ 0, & \text{o.w.} \end{cases}$

$f_{j\xi}$: random variable

$j = 1, \cdots, N$, $\xi = 1, \cdots, N$



$$\sum_{j=1}^{N} f_{j\xi} = 1$$

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- $E(f_{j\xi}) = P(f_{j\xi} = 1) = \frac{(N-1)!}{N!} = \frac{1}{N}$

$$E(f_{j\xi} f_{j'\xi'}) = P(f_{j\xi} = 1, f_{j'\xi'} = 1) = \begin{cases} \frac{1}{N}, & j = j',\ \xi = \xi' \\ 0, & j = j',\ \xi \neq \xi' \\ 0 & j \neq j',\ \xi = \xi' \\ \frac{(N-2)!}{N!} = \frac{1}{N(N-1)}, & j \neq j',\ \xi \neq \xi' \end{cases}$$

- Let the response contributed by the plot originally labelled by ξ be denoted by μ_ξ, and let the plot receive the label j, then define

$$m_j = \sum_{\xi=1}^{N} f_{j\xi} \mu_\xi$$

- Let $\bar{\mu} = \frac{1}{N} \sum_{\xi=1}^{N} \mu_\xi$, $\mu_\xi = \bar{\mu} + (\mu_\xi - \bar{\mu})$, then

$$m_j = \sum_{\xi=1}^{N} f_{j\xi} [\bar{\mu} + (\mu_\xi - \bar{\mu})] = \bar{\mu} + \underbrace{\sum_{\xi=1}^{N} f_{j\xi} (\mu_\xi - \bar{\mu})}_{\eta_j}$$

η_j: a random variable; called "unit error" or "plot error". It's the error represents the variation caused by randomization

- $E(\eta_j) = \sum_{\xi=1}^{N} (\mu_\xi - \bar{\mu}) E(f_{j\xi}) = 0$

$E(\eta_j \eta_{j'}) =$

$$E(\eta_j\eta_{j'})=\sum_{\xi_1=1}^{N}\sum_{\xi_1'=1}^{N}(\mu_{\xi_1}-\bar{\mu})(\mu_{\xi_1'}-\bar{\mu})E(f_{j\xi_1}f_{j'\xi_1'})$$

$$Var(\eta_j)=\frac{1}{N}\sum_{\xi_1=1}^{N}(\mu_{\xi_1}-\bar{\mu})^2=\frac{N-1}{N}\cdot\sigma_v^2=(1-\frac{1}{N})\sigma_v^2$$

$$cov(\eta_j,\eta_{j'})=-\frac{1}{N}\underbrace{\boxed{\frac{1}{(N-1)}\sum_{\xi_1=1}^{N}(\mu_{\xi_1}-\bar{\mu})^2}}_{\sigma_v^2}=-\frac{1}{N}\sigma_v^2$$

$\sigma_v^2 \leftarrow$ variation of plots

- $m_j=\bar{\mu}+\eta_j$

  $E(m_j)=\bar{\mu}$

  $cov(m_j,m_{j'})=(\delta_{jj'}-\frac{1}{N})\sigma_v^2$, $\delta_{jj'}=\begin{cases}1, & j=j'\\ 0, & o.w.\end{cases}$

- It should be noted that when observing the response, any observation (i.e. y's) may be affected by a "technical error". Denote the error by e_j. Assume that $E(e_j)=0$, $Var(e_j)=\sigma_e^2$, $cov(e_j,e_{j'})=0$ if $j\neq j'$

- Then, write the linear model

$$y_j=\tau_{T(j)}+m_j+e_j$$

$\tau_{T(j)}$ ← design. m_j ← caused by plan

fixed but unkown constants

$$=\boxed{\tau_{T(j)}+\bar{\mu}}+\boxed{\eta_j+e_j}$$

random variable

$$=\quad \boldsymbol{\tau}\quad+\quad \boldsymbol{Z}$$

→ c.f. the linear model in LNp.2-10

$$(*)\cdots\begin{cases}E(\boldsymbol{Y})=\boldsymbol{\tau}+\bar{\mu}\\ cov(y_j,y_{j'})=\begin{cases}\underbrace{\boxed{(1-\frac{1}{N})\sigma_v^2+\sigma_e^2}}_{=\sigma^2}, & j=j'\\ \underbrace{\boxed{-\frac{1}{N}\sigma_v^2}}_{=\rho\sigma^2}, & j\neq j'\end{cases}\Rightarrow\sigma^2\begin{bmatrix}1 & & \rho\\ & \ddots & \\ \rho & & 1\end{bmatrix}\end{cases}$$

- An alternative to get the model $(*)$

  Recall: LNp2-10, $\boldsymbol{Y}=(\mu\boldsymbol{1}+\boldsymbol{\tau})+\underbrace{(\boldsymbol{Z}-\mu\boldsymbol{1})}_{=\boldsymbol{Z}'}$, $\boldsymbol{Z}'\Rightarrow E(\boldsymbol{Z}')=0$, $Var(\boldsymbol{Z}')=\sigma^2 I$

  - If μ is regarded as a random effect

$$\boldsymbol{Y}=\boldsymbol{\tau}+\overbrace{(\mu+\boldsymbol{Z}')}^{\equiv\boldsymbol{Z}''}$$

$$E(\boldsymbol{Z}'')=0$$

$$Var(\mu+Z''_\alpha)=Var(\mu)+Var(Z''_\alpha)$$

$$cov(\mu+Z''_\alpha,\mu+Z''_\beta)=Var(\mu)$$

  - Recall: when μ is regarded as fixed effect

    $\Rightarrow$ μ confounded with $\boldsymbol{\tau}$

    $\Rightarrow$ the reason why we are often not interested in T_0 and seperate it from T_F as in LNp.2-21

(Bailey) Sec. 2-13. A more general model (randomization model / random effect model)

$$\mathbb{Y} = \mathbb{T} + \mathbb{Z} \quad (\text{LN p.2-10, } cov(\mathbb{Z}) = \sigma^2 I)$$

- Here, we change the assumption to

$$cov(Z_\alpha, Z_\beta) = \begin{cases} \sigma^2, & \text{if } \alpha = \beta \\ \rho\sigma^2, & \text{if } \alpha \neq \beta \end{cases}$$

$\rho \to cor(Z_\alpha, Z_\beta)$

where ρ may not be zero ($\Rightarrow \sigma^2 I$ is a special case)

- model: $E(\mathbb{Y}) = \mathbb{T}$

$$cov(\mathbb{Y}) = \rho\sigma^2 \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{bmatrix} + (1-\rho)\sigma^2 \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$$

$$= \rho\sigma^2 \cdot J + (1-\rho)\sigma^2 \cdot I$$

$$= \sigma^2[\rho J + (1-\rho) I]$$

- eigenvalue, eigenvector, eigenspace of $cov(\mathbb{Y})$
  - $\because I u_0 = u_0$, $J \cdot u_0 = N \cdot u_0$, $u_0 \to \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$

    $\therefore cov(\mathbb{Y}) \cdot u_0 = \sigma^2[(1-\rho) + N\rho] \cdot u_0$

    $\Rightarrow u_0$ is an eigenvector of $cov(\mathbb{Y})$ with eigenvalue

  - Let $x \in V_0^\perp$ (i.e. $x \in V$ and $x \cdot u_0 = 0$)

    Then $J \cdot x = 0$, $\therefore cov(\mathbb{Y}) \cdot x = \sigma^2(1-\rho) \cdot x$

    $\Rightarrow x$ is an eigenvector of $cov(\mathbb{Y})$ with eigenvalue
  - V_0 is the eigenspace corresponding to eigenvalue $\sigma^2[(1-\rho)+\rho N]$

    $V_0^\perp$: : : : : : $\sigma^2(1-\rho)$
  - $V = \underbrace{V_0}_{dim=1} \oplus \underbrace{V_0^\perp}_{dim=N-1}$

- changing $cov(\mathbb{Y})$ from $\sigma^2 I$ to $\sigma^2[(1-\rho)I + \rho J]$,

  (i) make no difference to $E[x \cdot \mathbb{Y}]$ $\leftarrow$ no change in the 1st moment of $\mathbb{Y}$

  (ii) but, it changes the expection of quadratic function of $\mathbb{Y}$.

  such as $Var(x \cdot \mathbb{Y})$, SS, MS,

  Q: How does it change?

  - If $cov(\mathbb{Y}) = \sigma^2 I$, $\Rightarrow$ For any $x \in V$, $cov(\mathbb{Y}) \cdot x = \sigma^2 I \cdot x = \sigma^2 \cdot x$
  - If x is an eigenvector of $cov(\mathbb{Y})$ with eigenvalue ξ_p

    $cov(\mathbb{Y}) \cdot x = \xi_p \cdot x$

  - careful replacement of σ^2 by ξ_p give the correct generalized results of Thm 2.4, 2.5, 2.6, 2.9.
  - one possible difficult, $x \_ \xi_1$, $z \_ \xi_2$

Theorem 2.10 *Suppose that* $\mathbb{E}(\mathbf{Y}) = \tau \in V_T$ *and* $\mathrm{Cov}(\mathbf{Y}) = \mathbf{C}$. *Then the following hold.*

may not be $\sigma^2 I$.

c.f. Thm2.4(i)

(i) *If* W *is any subspace of* V *then* $\mathbb{E}(P_W(\mathbf{Y})) = P_W\tau$.

a linear transformation

$= P_W(E(Y))$

(ii) *If* W *consists entirely of eigenvectors of* $\mathbf{C}$ *with eigenvalue* ξ, *and if* $\dim W = d$, *then*

c.f. Thm2.4(ii)

same eigenvalue

$$\mathbb{E}(\|P_W(\mathbf{Y})\|^2) = \|P_W(\tau)\|^2 + d\xi.$$

$\leftrightarrow \sigma^2$ in Thm 2.4 (ii)

(iii) *If* $\mathbf{x} \in V_T$ *and* $\mathbf{x}$ *is an eigenvector of* $\mathbf{C}$ *then the best linear unbiased estimator of* $\mathbf{x}\cdot\tau$ *is* $\mathbf{x}\cdot\mathbf{Y}$.

c.f. Thm2.5(i)

c.f. Thm2.5(ii)

(iv) *If* $\mathbf{x}$ *is an eigenvector of* $\mathbf{C}$ *with eigenvalue* ξ *then the variance of* $\mathbf{x}\cdot\mathbf{Y}$ *is* $\|\mathbf{x}\|^2\xi$. $\leftrightarrow \sigma^2$ in Thm2.5(ii)

(v) *Suppose that* $\mathbf{x}$ *and* $\mathbf{z}$ *are eigenvectors of* $\mathbf{C}$ *with eigenvalues* ξ *and* η *respectively. If* $\xi = \eta$ *then* $\mathrm{cov}(\mathbf{x}\cdot\mathbf{Y}, \mathbf{z}\cdot\mathbf{Y}) = (\mathbf{x}\cdot\mathbf{z})\xi$; *if* $\xi \neq \eta$ *then* $\mathrm{cov}(\mathbf{x}\cdot\mathbf{Y}, \mathbf{z}\cdot\mathbf{Y}) = 0$.

c.f. Thm2.5 (iii)

$\because$ $x\cdot z = 0$ if they belong to different eigenspaces

(vi) *Suppose that* $\mathbf{x}$ *is an eigenvector of* $\mathbf{C}$ *with eigenvalue* ξ, *that* $\mathbf{x} = \sum_i(\lambda_i/r_i)\mathbf{u}_i$, *that* W *is a d-dimensional subspace consisting of eigenvectors of* ξ *orthogonal to* V_T. *If* $\mathbf{Y}$ *has a multivariate normal distribution then*

c.f. Thm2.6(i)

residual space

$$\frac{\mathbf{x}\cdot\mathbf{Y} - \sum\lambda_i\tau_i}{\sqrt{\left(\sum\frac{\lambda_i^2}{r_i}\right)\times \mathrm{MS}(W)}}$$

$= x\cdot\tau$ $\quad x\cdot\tau = \sum_{i=1}^{t}\lambda_i\tau_i$

$\to \hat{\sigma}^2$ for some σ^2

has a t*-distribution on d degrees of freedom and* $\mathrm{SS}(W)/\xi$ *has a* χ^2*-distribution on d degrees of freedom.*

σ^2 in Thm2.6(i)

(vii) *If* W_1 *and* W_2 *are subspaces with dimensions* d_1 *and* d_2, *both consisting of eigenvectors of* $\mathbf{C}$ *with eigenvalue* ξ, *orthogonal to each other, with* $P_{W_1}\tau = P_{W_2}\tau = \mathbf{0}$, *and if* $\mathbf{Y}$ *has a multivariate normal distribution then* $\mathrm{MS}(W_1)/\mathrm{MS}(W_2)$ *has an* F*-distribution on* d_1 *and* d_2 *degrees of freedom.*

in same stratum.

null hypothesis

within stratum comparison

Definition A *stratum* is an eigenspace of $\mathrm{Cov}(\mathbf{Y})$ (note that this is not the same as a stratum in sampling).

- ANOVA table

cannot test $\bar{\tau}=0$

stratum	source	df	EMS
V_0 'mean'	mean	1	$N\bar{\tau}^2+\xi_0$
$V_0^{\perp}$ 'plots'	treatments	$t-1$	$\boxed{\frac{\sum_i r_i(\tau_i-\bar{\tau})^2}{t-1}+\xi_1}$
	residual	$N-t$	$\boxed{\xi_1}$
Total		N	

V_0: dim=1, eigenvalue $=\sigma^2[(1-\rho)+N\rho]$

$V_0^{\perp}$: dim=N-1, eigenvalue$=\sigma^2(1-\rho)$

Thm2.10(ii)

V (dim=N)

(mean structure) treatment structure	plot structure (Variance-Covariance structure)
overall mean space — V_0	V_0 — mean stratum
V_T: $V_0 \oplus W_T$	$\oplus$
treatment space — W_T	$V_0^{\perp}$ — plot stratum ($W_T \oplus V_T^{\perp}$)
residual space — $V_T^{\perp}$