- To estimate $\underset{x\cdot\tau}{\mathrm{Var}(x\cdot Y)} = \left(\sum_{i=1}^{t} \frac{\lambda_i^2}{r_i}\right)\sigma^2$ unknown parameter

Note that $\underset{\hat{\sigma}^2}{\underset{|||}{\text{MS(residual)}}}$ is an unbiased est'or of σ^2

$\therefore \left(\sum_{i=1}^{t} \frac{\lambda_i^2}{r_i}\right)\hat{\sigma}^2$ is an unbiased est'or of $\mathrm{Var}(x\cdot Y)$

Def: ① standard error of $\sum\lambda_i\hat{\tau}_i$ ($\to \sum\lambda_i\tau_i$) is

$$\sqrt{\sum\frac{\lambda_i^2}{r_i}}\cdot\hat{\sigma}$$

② the std. error for $\hat{\tau}_i$ is $\hat{\sigma}/\sqrt{r_i}$, which is referred to as the std. error of a mean, and is abbreviated as s.e.m

③ the std. error of $\hat{\tau}_i - \hat{\tau}_j$ is

$$\sqrt{\frac{1}{r_i}+\frac{1}{r_j}}\cdot\hat{\sigma}$$

called std. error of a difference, abbreviated as s.e.d.

Theorem 2.6 *Suppose that the distribution of* $\mathbf{Y}$ *is multivariate normal, that* $\mathbb{E}(\mathbf{Y}) = \tau \in V_T$ *and that* $\mathrm{Cov}(\mathbf{Y})$ *is a scalar matrix. Then the following hold.* add this for test

(i) *If* $\mathbf{x} = \sum_{i=1}^{t}(\lambda_i/r_i)\mathbf{u}_i$ *then*

$$\frac{\mathbf{x}\cdot\mathbf{Y} - \overset{x\cdot\tau}{\sum\lambda_i\tau_i}}{\sqrt{\left(\sum\frac{\lambda_i^2}{r_i}\right)\times \underbrace{\text{MS(residual)}}_{\hat{\sigma}^2}}}$$

has a t*-distribution on* $\underset{\dim(V_T^\perp)}{N-t}$ *degrees of freedom.*

(ii) *If* $\mathbf{x}$ *and* $\mathbf{z}$ *are in* V *and* $\mathbf{x}\cdot\mathbf{z} = 0$ *then* $\mathbf{x}\cdot\mathbf{Y}$ *and* $\mathbf{z}\cdot\mathbf{Y}$ *are independent estimators.*

(Bailey) Sec. 2.9. replication ($r_1, \dots, r_t$)

- Assume that all estimates of simple treatment differences ($\hookrightarrow \tau_i - \tau_j$) are equally important. $\Rightarrow \mathrm{Var}(\hat{\tau}_i - \hat{\tau}_j) = \sigma^2\left(\frac{1}{r_i}+\frac{1}{r_j}\right)$
- average variance of these est'ors:

$$\frac{1}{\binom{t}{2}}\sum_{i\neq j}\mathrm{Var}(\hat{\tau}_i - \hat{\tau}_j) \propto \frac{1}{t(t-1)}\sum_{i=1}^{t}\sum_{j\neq i}\left(\frac{1}{r_i}+\frac{1}{r_j}\right) = \frac{1}{t}\sum_{i=1}^{t}\frac{1}{r_i}.$$

Note: no parameter in it.

Proposition 2.7 *If positive numbers* $r_1, \dots, r_t$ *have a fixed sum* R *then* $\sum(1/r_i)$ *is minimized when* $r_1 = r_2 = \cdots = r_t = R/t$. $\to$ may not be an integer

Recall: Thm2.5(i) BLUE for $x\cdot\tau$ is $x\cdot Y$

e.g. $\underset{\downarrow}{A}$, $\underset{\downarrow}{B}$: two treatments.
$\tau_A \quad \tau_B$

$n = r_A + r_B$

$$E(Y) = \begin{array}{c} r_A\{ \\ \\ r_B\{ \end{array}\underbrace{\begin{bmatrix} 0 & 1 \\ \vdots & \vdots \\ 0 & 1 \\ 1 & 0 \\ \vdots & \vdots \\ 1 & 0 \end{bmatrix}}_{X}\begin{bmatrix}\beta_0 \\ \beta_1\end{bmatrix} \Rightarrow \overset{\text{BLUE}}{\hat{\beta}} = (X'X)^{-1}(X'Y) = \begin{bmatrix} 1/r_B & 0 \\ 0 & 1/r_A \end{bmatrix}\cdot\begin{bmatrix} 0 - 0\ 1 - 1 \ (u_B) \\ 1 - 1\ 0 - 0 \ (u_A) \end{bmatrix}\begin{bmatrix} y \end{bmatrix}$$

$\parallel$ same vector

$$= \begin{bmatrix} SUM_{T=B}/r_B \\ SUM_{T=A}/r_A \end{bmatrix} \equiv \begin{bmatrix} \hat{\tau}_B \\ \hat{\tau}_A \end{bmatrix}$$

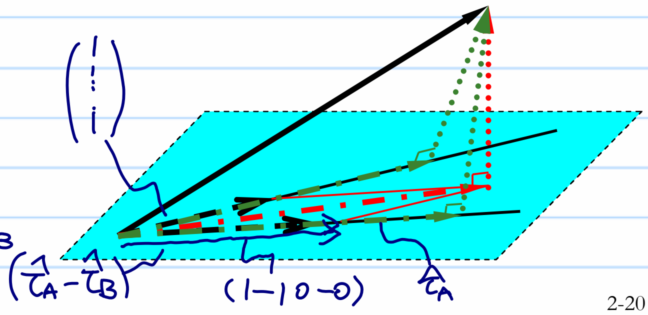
different est'ors

WHY?

$$E(Y) = \begin{bmatrix} 1 & 1 \\ \vdots & \vdots \\ 1 & 1 \\ 1 & 0 \\ \vdots & \vdots \\ 1 & 0 \end{bmatrix}\begin{bmatrix}\beta_0 \\ \beta_1\end{bmatrix} \Rightarrow \overset{\text{BLUE}}{\hat{\beta}} = (X'X)^{-1}X'Y = \frac{1}{r_A r_B}\begin{bmatrix} r_A & -r_A \\ -r_A & n \end{bmatrix}\begin{bmatrix} n\cdot\bar{y} \\ r_A\cdot\hat{\tau}_A \end{bmatrix} = \begin{bmatrix} \hat{\tau}_B \\ \hat{\tau}_A - \hat{\tau}_B \end{bmatrix}$$

$$E(Y) = \begin{cases} \beta_0, & T = B \\ \beta_1, & T = A \end{cases} \Rightarrow \begin{array}{l} \beta_0 = \tau_B \\ \beta_1 = \tau_A \end{array}$$

$$E(Y) = \begin{cases} \beta_0 + \beta_1, & T = A \\ \beta_0, & T = B \end{cases} \Rightarrow \begin{array}{l} \beta_0 = \tau_B \\ \beta_1 = \tau_A - \tau_B \end{array}$$



(Bailey) Sec 2.10. Allowing for the overall mean

- In LNp.2-16,

$$EMS(V_T) = E\left(\frac{\|P_{V_T}Y\|^2}{\underset{t}{\dim(V_T)}}\right) = \frac{\sum_{i=1}^{t} r_i\tau_i^2}{t} + \frac{\cancel{t}\sigma^2}{\cancel{t}} \quad \sigma^2 = EMS(\text{residual})$$

r_i: # of replicates in treatment i

$$(*) \text{ --- } EMS(V_T) - EMS(\text{residual}) = \frac{\sum_{i=1}^{t} r_i\tau_i^2}{t} \geq 0$$

$(*) = 0$ if and only if $\tau_1 = \tau_2 = \cdots = \tau_t = 0$

$\hookrightarrow$ usually do not measure on a scale that make this plausible

- However, it is often plausible that $\tau_1 = \tau_2 = \cdots = \tau_t$,

i.e. $E(Y_\omega) = \kappa$ for all $\omega \in \Omega$ ($\kappa \to$ an unknown constant (parameter))

$= \kappa\, u_0$, $u_0 \to \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$

the model is called null model

- Let V_0 be the subspace of V which consists of scalar multiple of u_0 $\Rightarrow \dim(V_0) = 1$

$\Rightarrow \{u_0\}$ is a basis of V_0.

• By Thm 2.3 (VII) (LN p.2-9)

Proposition 2.8 *If* $\mathbf{v} \in V$ *then*

$$P_{V_0}\mathbf{v} = \left(\frac{\mathbf{v}\cdot\mathbf{u}_0}{\mathbf{u}_0\cdot\mathbf{u}_0}\right)\mathbf{u}_0 = \left(\frac{\text{grand total of }\mathbf{v}}{N}\right)\mathbf{u}_0 = \bar{v}\mathbf{u}_0 \quad and \quad \|P_{V_0}\mathbf{v}\|^2 = \left(\frac{\text{grand total of }\mathbf{v}}{N}\right)^2 \underbrace{\mathbf{u}_0\cdot\mathbf{u}_0}_{=N=(\#\text{ of plots})} = \frac{(\text{grand total of }\mathbf{v})^2}{N} = N\bar{v}^2.$$

$$r_i\left\{\begin{bmatrix}1\\ \vdots\\ 1\\ 0\\ \vdots\\ 0\end{bmatrix}\right.$$

• ∵ $u_0 = u_1 + \cdots + u_t \in V_T \Rightarrow V_0 \subset V_T$ (Recall: $V = \underbrace{V_T}_{\text{treatment}} \oplus \underbrace{V_T^{\perp}}_{\text{residual}}$) ← w.r.t. V

We can define

$W_T = \{v \in V_T \mid v \text{ is orthogonal to } V_0\} = V_T \cap V_0^{\perp}$ ← w.r.t. V

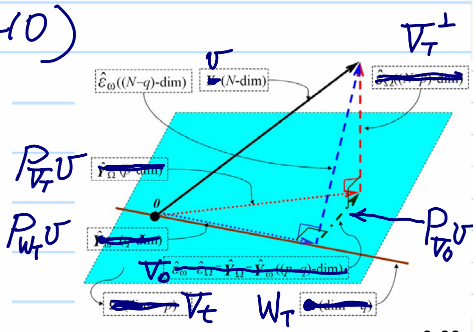
$v \in W_T$
$v \cdot u_0 = v_1 + \cdots + v_N = 0$
treatment contrast

and find that (by Thm 2.3, LN p.2-10)

(i) $\dim W_T = \dim V_T - \dim V_0 = t - 1$;

(ii) $P_{W_T}\mathbf{v} = P_{V_T}\mathbf{v} - P_{V_0}\mathbf{v}$ for all $\mathbf{v}$ in V;

(iii) $\|P_{W_T}\mathbf{v}\|^2 + \|P_{V_0}\mathbf{v}\|^2 = \|P_{V_T}\mathbf{v}\|^2$ for all $\mathbf{v}$ in V.



• Applying (ii) and (iii) with $\mathbf{v} = \tau$ gives $\begin{bmatrix}\tau_1\\ \tau_1\\ \tau_2\\ \vdots\end{bmatrix} = \sum_{i=1}^{t} \tau_i \cdot u_i \in V_T$, $\tau = E[Y]$

$$P_{W_T}\tau \overset{\text{by (ii)}}{=} \underbrace{\tau}_{P_{V_T}\tau} - P_{V_0}\tau = \tau - \bar{\tau}\mathbf{u}_0 = \sum_{i=1}^{t}(\tau_i - \bar{\tau})\mathbf{u}_i$$

$\mathbf{u}_0 = u_1 + \cdots + u_t$, $\quad \bar{\tau} = \frac{\sum r_i \tau_i}{N} \neq (\tau_1 + \cdots + \tau_t)/t$

$$\sum_{i=1}^{t} r_i(\tau_i - \bar{\tau})^2 \underset{\because\ u_i\text{'s is a basis of } V_T}{=} \|P_{W_T}\tau\|^2 \overset{\text{by (iii)}}{=} \|\tau\|^2 - \|\bar{\tau}\mathbf{u}_0\|^2 = \sum_{i=1}^{t} r_i\tau_i^2 - N\bar{\tau}^2,$$

which is zero if and only if all the τ_i are equal.

• Do not know τ:

Applying (ii) with $\mathbf{v} = \mathbf{y}$ gives ← contain information of τ

$$P_{W_T}\mathbf{y} \overset{\text{by (ii)}}{=} \underbrace{\sum_{i=1}^{t}\left(\frac{\text{sum}_{T=i}}{r_i}\right)\mathbf{u}_i}_{P_{V_T}y} - \underbrace{\frac{\text{sum}}{N}\mathbf{u}_0}_{P_{V_0}y}$$

$\mathbf{u}_0 = u_1 + \cdots + u_t$

= fitted values for treatments − fit for null model

$$= \sum_{i=1}^{t}(\hat{\tau}_i - \bar{y})\mathbf{u}_i.$$

Know $\hat{\tau}_1 - \bar{y}, \hat{\tau}_2 - \bar{y}, \ldots, \hat{\tau}_t - \bar{y} \Rightarrow$ do not know $\hat{\tau}_1, \hat{\tau}_2, \ldots, \hat{\tau}_t$

The coefficients $\hat{\tau}_i - \bar{y}$ are called *treatment effects*. Taking sums of squares gives

$$\|P_{W_T}\mathbf{y}\|^2 \overset{\text{by (iii)}}{=} \underbrace{\sum_{i=1}^{t}\frac{\text{sum}^2_{T=i}}{r_i}}_{\sum_{i=1}^{t} r_i\hat{\tau}_i^2} - \underbrace{\frac{\text{sum}^2}{N}}_{=N\bar{y}^2}.$$

c.f.

Def: (Recall: LNp.2-14, c.f. LNp.2-16)

- sum of square for W_T: called SS for treatments abbreviated as SS(treatment)

$$MS(\text{treatment}) = SS(\text{treatment}) / \dim(W_T) = t-1$$

- SS for V_0: called (crude) SS for mean abbreviated as SS(mean)

$$MS(\text{mean}) = SS(\text{mean}) / \dim(V_0) = 1$$

- $SS(\text{treatment}) = \text{crude SS for treatment } (\|P_{V_T} y\|^2) - SS(\text{mean})$

(Bailey) 2.11. hypothesis testing.

According to "treatment structure", we split

$$V = \underbrace{V_T}_{\text{treatment}} \oplus \overbrace{V_T^{\perp}}^{\text{residual}}$$

	V	$=$	V_0	$\oplus$	W_T	$\oplus$	$V_T^{\perp}$
dimension degree of freedom	N	$=$	1	$+$	$(t-1)$	$+$	$(N-t)$
data	$\mathbf{y}$	$=$	$\underbrace{\bar{y}\mathbf{u}_0}_{= P_{V_0} y}$	$+$	$\underbrace{\left(\sum_i \text{mean}_{T=i}\mathbf{u}_i - \bar{y}\mathbf{u}_0\right)}$	$+$	$\underbrace{\left(\mathbf{y} - \sum_i \text{mean}_{T=i}\mathbf{u}_i\right)}$
			overall mean		treatment effects $= P_{W_T} y$		residual $= P_{V_T}^{\perp} y$
sum of squares	$\sum_{\omega\in\Omega} Y_\omega^2$	$=$	$\underbrace{\frac{\text{SUM}^2}{N}}_{= \|P_{V_0} y\|^2}$	$+$	$\underbrace{\text{SS(treatments)}}_{= \|P_{W_T} y\|^2}$	$+$	$\underbrace{\text{SS(residual)}}_{= \|P_{V_T}^{\perp} y\|^2}$
mean square $= SS/\dim(\cdot) \to$ d.f.			$\frac{\text{SUM}^2}{N}$		$\frac{\text{SS(treatments)}}{t-1}$		$\frac{\text{SS(residual)}}{N-t}$
expected mean square $= E\left(\frac{\|P_\cdot y\|^2}{\dim(\cdot)}\right)$			$N\bar{\tau}^2 + \sigma^2$		$\frac{\sum_i r_i \tau_i^2 - N\bar{\tau}^2}{t-1} + \sigma^2$		σ^2

- Test $H_0: \tau = 0$ against $H_1: \tau \neq 0$

$$V = V_0 \oplus W_T \oplus V_T^{\perp}$$

	V_0	W_T	$V_T^{\perp}$
expected mean square	$\underbrace{N\bar{\tau}^2+\sigma^2}_{\parallel}$	$\underbrace{\frac{\sum_i r_i\tau_i^2 - N\bar{\tau}^2}{t-1}+\sigma^2}_{\parallel}$	$\underbrace{\sigma^2}_{\parallel}$
under H_0	σ^2	$\frac{\sum r_i \tau_i^2}{t-1}+\sigma^2$	σ^2

MS(mean) $\approx$ MS(residual) $\Rightarrow$ $\frac{\text{MS(mean)}}{\text{MS(residual)}}$ small $\to H_0$

under H_1 $\gg$ large $\to H_1$

- $H_0: \tau_1 = \cdots = \tau_t \equiv K$ against H_1: at least one not equal

$$V = V_0 \oplus W_T \oplus V_T^{\perp}$$

	V_0	W_T	$V_T^{\perp}$
expected mean square	$N\bar{\tau}^2+\sigma^2$	$\frac{\sum_i r_i\tau_i^2 - N\bar{\tau}^2}{t-1}+\sigma^2$	σ^2
under H_0	$NK^2+\sigma^2$	σ^2	σ^2

MS(treatment) $\approx$ MS(residual)] $\to$ $\frac{\text{MS(treatment)}}{\text{MS(residual)}}$ small $-H_0$

under H_1 $\gg$ large $-H_1$

- ANOVA table

$\hookrightarrow$ interested in the comparison of treatment means (i.e., τ_i's)

source	sum of squares	degrees of freedom	mean square	variance ratio
~~mean~~	$\frac{\text{sum}^2}{N}$	1	SS(mean)	$\frac{\text{MS(mean)}}{\text{MS(residual)}}$
treatments	$\sum_i \frac{\text{sum}^2_{T=i}}{r_i} - \frac{\text{sum}^2}{N}$	$t-1$	$\frac{\text{SS(treatments)}}{t-1}$	$\frac{\text{MS(treatments)}}{\text{MS(residual)}}$
residual	← . . . . . by subtraction . . . . . →		$\frac{\text{SS(residual)}}{\text{df(residual)}}$	under H_0 $F_{1,N-t}$
Total	$\sum_\omega y_\omega^2$	N		under H_0 $F_{t-1,N-t}$

$\sum_{\omega\in\Omega}(y_\omega - \bar{y})^2$

$V = V_0 \oplus W_T \oplus V_T^{\perp}$

$V_0^{\perp} = W_T \oplus V_T^{\perp}$

Theorem 2.9 *Suppose that the distribution of* $\mathbf{Y}$ *is multivariate "normal". Let* W_1 *and* W_2 *be subspcaces of* V *with dimensions* d_1 *and* d_2. *Then the following hold.*

(i) *If* $\overbrace{P_{W_1}\tau = \mathbf{0}}^{H_0}$ *then* $\text{SS}(W_1)/\sigma^2$ *has a* χ^2*-distribution with* d_1 *degrees of freedom.* ($W_1 \to V_0$)

(ii) *If* W_1 *is orthogonal to* W_2 *and* $P_{W_1}\tau = P_{W_2}\tau = \mathbf{0}$ *then* $\text{MS}(W_1)/\text{MS}(W_2)$ *has an* F*-distribution with* d_1 *and* d_2 *degrees of freedom.*

H_1 $(\tau_1, \cdots, \tau_t) \in \mathbb{R}^t$ $\quad \mathbb{E}(Y_\omega) = \tau_{T(\omega)} \qquad \mathbb{E}(\mathbf{Y}) \in V_T \qquad V_T$

add assumption $\tau_1 = \tau_2 = \cdots = \tau_t$ $\qquad \cup \qquad \Big\} \; W_T$

H_0 $(\tau_1, \cdots, \tau_t) \in \mathbb{R}^1$ $\quad \mathbb{E}(Y_\omega) = \kappa \qquad \mathbb{E}(\mathbf{Y}) \in V_0 \qquad V_0$

H_0 H_a

add assumption $\tau_1 = 0$ $\qquad \cup \qquad \Big\} \; V_0$

$(\tau_1, \cdots, \tau_t) \in \mathbb{R}^0$ $\quad \mathbb{E}(Y_\omega) = 0 \qquad \mathbb{E}(\mathbf{Y}) \in \{\mathbf{0}\} \qquad \{\mathbf{0}\}$

Sec 2.12. Sufficient replication for power
(reading assignment)