Design Issues In Fractional Factorial Split-Plot Experiments

-Bingham, D., and Sitter, R.R. (2001)

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Outline

- Induction
- Introduction
  - -Notation
  - -Wood Product example
- Fractional Factorial Split-Plot Designs
  - -Analysis
- Return to Wood Product example
- Do exercise

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Induction

It is often impractical to perform the experimental runs of a fractional factorial in a completely random order. In these cases, restrictions on the randomization of the experimental trials are imposed and the design is said to have a split-plot structure. Similar to fractional factorials, the "goodness" of fractional factorial split-plot designs can be judged using the minimum aberration criterion. However, the split-plot nature of the design implies that not all factorial effects can be estimated with the same precision. In this paper, we discuss the impact of the randomization restrictions on the design. We show how the split-plot structure affects estimation, precision, and the use of resources. We demonstrate how these issues affect design selection in a real industrial experiment.

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Introduction

- Notation:

(1) FF design=fractional factorial design

(2) FFSP design= fractional factorial split-plot design

(3) WP factor=whole-plot factor , with k_1 factors

use Capital letter (e.g. A,B,C,…)

(4) SP factor=sub-plot factor, with k_2 factors

use Lowercase letter (e.g. p,q,r,…)

(5) MA=minimum aberration

Introduction (cont.)

- Q: Why need FFSP design??
- Ans:

When it is expensive or difficult to change the levels of some of the factors,or because of actual physical restrictions on the process, it can be impractical to perform experimental runs in a completely random order. In such cases, restrictions on the randomization of experimental trials are imposed resulting in a split-plot structure

Introduction (cont.)

- Q: A question which commonly arises is which of the many possible FFSP designs should be used??
- Ans:

minimum aberration (MA) FFSP designs (Huang, Chen and Voelkel (1998);Bingham and Sitter (1999)). However, unlike FF designs, FFSP designs have two sources of error which are used to assess the significance of the effects.

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Wood Product Example

- Object:

  Investigate the affect of certain factors on the swelling properties of a wood product.

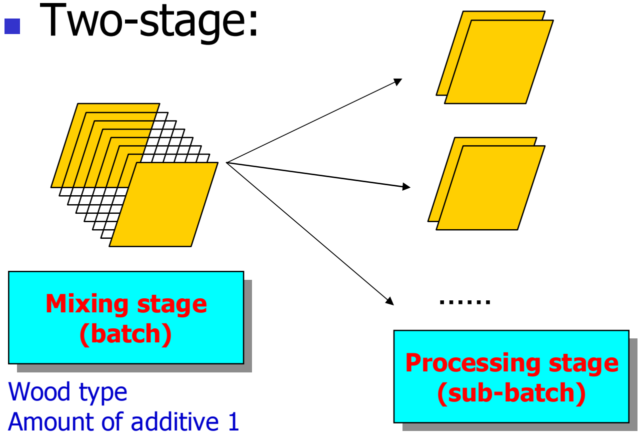
- Note:

(1) $k_1=5$ => 5 WP factors
$k_2=3$ => 3 SP factors
Each of the $k_1 + k_2 = 8$ factors are to be investigated at two levels.

(2) To construct the design we assign p_1 WP factors and p_2 SP factors to interactions involving the remaining factors, where $p = p_1+p_2$ p fractional generators

Wood Product Example (cont.)

- Two-stage:



A - Wood type
B - Amount of additive 1
C - Amount of additive
D - Wood size
E - Wood moisture

p - Process time
q - Pressure
r - Material density.

$2^{(k_1+k_2)-(p_1+p_2)}$ FFSP design

- In Wood Product Example:

  $k_1=5, k_2=3, p_1=1, p_2=1$ (E=ABCD, r=pq)

- *Defining contrast sub-group*

  $I = pqr = ABCDE = ABCDEpqr$ (1)

- *Word length pattern*

  Let $A_i(D)$ denote the number of words of length i in the defining contrast sub-group, D, and let
  $W = (A3(D), A4(D), A5(D), \ldots)$ $W = (1, 0, 1, 0, 0, 1)$

- *Resolution*

  A design is the length of the shortest word in the defining contrast sub-group resolution III

- MA criterion

Suppose that *D*1 and *D*2 are $2^{(k_1+k_2)-(p_1+p_2)}$ FFSP design

If Ai(D1) = Ai(D2) for i = 3, ..., r − 1 and

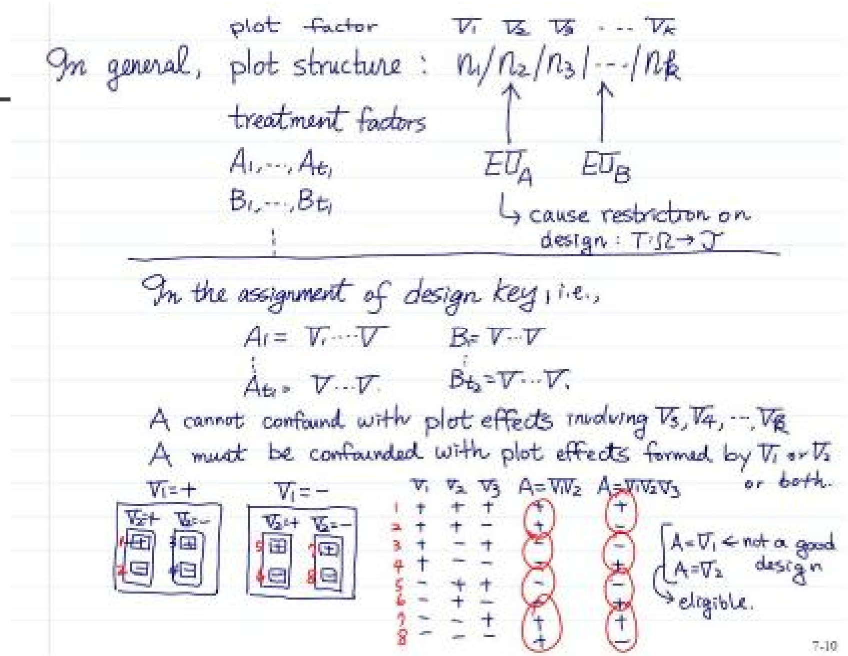
Ar(D1) < Ar(D2) we say D1 has less aberration than D2.

A design is said to be MA if no other design has less aberration.

Note that :
the MA criterion treats all factors and effects of the same order equally.

Discussion 1:

- 1.The approach to design construction used to get Equation (1) is too restrictive, resulting in a poor design.
  Though it is resolution V in the WP factors,
  it is only resolution III in the SP factors
- 2. r=AB => **OK!!**
  A=pq => **Does not permit!!**
  (see next slide)



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Drive Gears Example

we consider an experiment reported in Miller, Sitter,Wu, and Long (1993).

For the purpose of this illustration

We modify the example slightly and consider only those trials that correspond to

4 replicates of a 5 factor,16-run experiment.

Drive Gears Example (cont.)

A: furnace track *B:* tooth size C: part positioning
p: carbon potential *q:* operating mode

TABLE 1. Data for Gear Experiment.

A	B	C	p	q = ABCp	rep 1	rep 2	rep 3	rep 4
				|	15.0	16.5	20.0	22.5
			−	−	21.5	21.0	30.0	24.5
−	−	+	−	−	13.0	8.5	11.5	16.0
			−	+	−1.5	4.0	1.0	8.0
−	−	−	−	−	22.5	27.5	27.0	28.5
				|	13.0	14.0	19.0	14.5
		|		|	0.5	0.5	6.5	7.0
					3.5	4.5	4.0	9.0
					27.5	19.5	31.0	27.0
				|	17.0	14.0	18.0	17.5
−	−	+	−	+	17.5	11.5	10.0	1.0
			−	−	14.5	3.5	7.5	8.5
−	−	−	−	+	14.0	18.5	20.0	23.5
			−	−	22.0	25.0	26.0	14.0
		|			24.0	2.5	5.5	11.0
				|	13.5	7.0	3.0	10.5

K1=3

K2=2

P1=0

P2=1

(q=ABCp)

I=ABCpq

Modeling

$$y = f(WP\ effects) + \varepsilon + g(SP\ effects) + e$$

where ε and e are the WP and SP error terms, and f(.) and g(.) are functions of the WP and SP design parameters. It is assumed that ε and e are mutually independent random variables s.t.

$\varepsilon \sim N(0, \sigma^2_{\mathrm{WP}})$ and $e \sim N(0, \sigma^2_{\mathrm{SP}})$

Similar to the randomized block design, we expect the between batch variability to be larger than the within batch variability. That is, $\sigma^2_{\mathrm{WP}} > \sigma^2_{\mathrm{SP}}$

Discussion 2

- (1) df of WP error= $(l-1)2^{k_1-p_1}$ =24

  df of SP error = $(l-1)(2^{(k_1+k_2)-(p_1+p_2)} - 2^{k_1-p_1})$ =24

  where $l = replicates$

- (2) df of SP error $\geq$ df of WP error

  But $\sigma^2_{\mathrm{WP}} > \sigma^2_{\mathrm{SP}}$

  The different error terms and degrees of freedom imply that the power to detect significant effects is not the same for the WP and SP factors. Since the SP variability is assumed to be smaller than the WP variability and the SP error has at least as many degrees of freedom as the WP error, the power to detect significant SP effects is greater.

Discussion 2 (cont.)

- (3) How to distinguish treatment effect belongs which error term??
- Ans:

1. WP main effects and interactions involving only WP factors are compared to the WP error.
2. SP main effects or interactions that are aliased with WP main effects or interactions involving only WP factors are compared to the WP error.
3. SP main effects and interactions involving at least one SP factor that are not aliased with WP main effects or interactions involving only WP factors are compared to the SP error.

Discussion 2 (cont.)

- In Drive Gears Example (I=ABCpq)

A=BCpq
B=ACpq
C=ABpq
AB=Cpq
AC=Bpq
BC=Apq
pq=ABC

WP error term

p=ABCq
q=ABCp
Ap=BCq
Aq=BCp
Bp=ACq
Bq=Acp
Cp=ABq
Cq=ABp

SP error term

TABLE 2. ANOVA Table for Example 1

Effect	df	F Statistic	P-Value
A	1	1.72	0.2024
B	1	0.46	0.5026
C	1	80.85	0.0001
AB	1	0.09	0.7708
AC	1	2.88	0.1027
BC	1	0.01	0.9279
$pq = ABC$	1	0.00	0.9920
p	1	17.88	0.0003
q	1	25.57	0.0001
Ap	1	0.12	0.7307
Aq	1	6.96	0.0145
Bp	1	3.03	0.0945
Bq	1	0.26	0.6120
Cp	1	0.01	0.9348
Cq	1	3.03	0.0945

Discussion 2 (cont.)

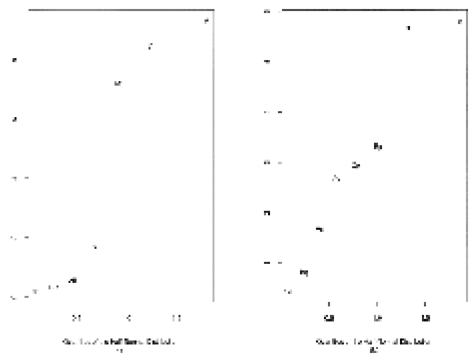
- (4) Testing significant

-half-normal

This paper use half-normal

-Lenth's method

WP effect and SP effect separately



Discussion 2 (cont.)

- (5)

Selecting a FFSP design not only does one have to consider the estimation issue captured by the MA criterion, but one will also wish to assess the significance of the factorial effects with as much precision as possible.

Such issues will be discussed in the next.

How to Choose a FFSP design

- Choosing Where to Split → K fixed =>how to choose k1 k2
- Choosing Where to Fractionate → K1 k2 fixed =>how to choose p1 p2
- Choosing Between Non-isomorphic MA design → K1 k2 p1 p2 the same =>how to choose design

K fixed
=>how to choose k1 k2

K1 k2 fixed
=>how to choose p1 p2

K1 k2 p1 p2 the same
=>how to choose design

K fixed
=>how to choose k1 k2

K1 k2 fixed
=>how to choose p1 p2

K1 k2 p1 p2 the same
=>how to choose design

Choosing Where to Split??

- 1.No choice of which factors should be WP factors and which should be SP factors.
  e.g., wood product example
- 2.Decision to run a split-plot experiment is driven by the cost of changing factor level settings.
  e.g., if factors *A* and *B* are hard-to-change factors, we might choose to make them WP factors
- In general, if the number of factors *k* is fixed, we can decrease the overall cost of experimentation by deciding to place more of our factors at the WP level,i.e. increasing *k*1. However, the end result will be a loss of power to detect the significance of these WP factors and their interactions. Need to trade off!!

K fixed
=>how to choose k1 k2

K1 k2 fixed
=>how to choose p1 p2

K1 k2 p1 p2 the same
=>how to choose design

Tables (to aid discussion)

- Table 3 :16 runs, k=6~10, FFSP design
- Table 4 :32 runs, k=6~10, FFSP design

- Example (table 3):

(A3,A4,A5,……)

$2^{(3+3)-(1+1)}$

$k_1.k_2.p_1.p_2$	WLP generators	WLP	# WP	$(a)^\dagger$	$(b)^\ddagger$
1.5.0.2	$Apqs$, $Aprt$	0 3	2	0	0
2.4.0.2*	$ABpr$, $Apqs$	0 3	4	1	0
2.4.0.2*	$ABpr$, $ABqs$	0 3	4	2	0
3.3.0.2*	$ABpq$, $ACpr$	0 3	8	3	0
3.3.1.1*	ABC, $Apqr$	1 1 1	4	0	0
4.2.1.1	$ABCD$, $ABpq$	0 3	8	1	0

The number of SP 2fi's tested against the WP error term.

C=AB, r=Apq

The number of SP 2fi's aliased with WP main effects.

Choosing Where to Fractionate??

- Example 2: 16 runs, k1=3,k2=3
  only have two FFSP design

(k1.k2.p1.p2)

*D*1 : $I = ABC = Apqr = BCpqr$ (3.3.1.1)
(C=AB,r=Apq)

*D*2 : $I = ABpq = ACpr = BCqr$ (3.3.0.2)
(q=Abp,r=ACp)

TABLE 3. Catalog of Minimum Aberration 16-Run FFSP Designs. (Designs are identified by $k_1.k_2.p_1.p_2$ and ordered by the number of factors $k = k_1 + k_2$.)

	$k_1.k_2.p_1.p_2$	WLP generators	WLP	# WP	$(a)^\dagger$	$(b)^\ddagger$
	1.5.0.2	$Apqs$, $Aprt$	0 3	2	0	0
	2.4.0.2*	$ABpr$, $Apqs$	0 3	4	1	0
	2.4.0.2*	$ABpr$, $ABqs$	0 3	4	2	0
D2	3.3.0.2*	$ABpq$, $ACpr$	0 3	8	3	0
D1	3.3.1.1*	ABC, $Apqr$	1 1 1	4	0	0
	4.2.1.1	$ABCD$, $ABpq$	0 3	8	1	0

*D*1 has resolution III.
*D*2 has resolution IV .
In generally,D2 is better than *D*1.

- Example 3: 32 runs, k1=4,k2=3 possible FFSP designs

*D*1 : *I* = *ABCD* = *ABpqr* = *CDpqr* (4.3.1.1)
(*D*=*ABC*,*r*=*ABpq*)

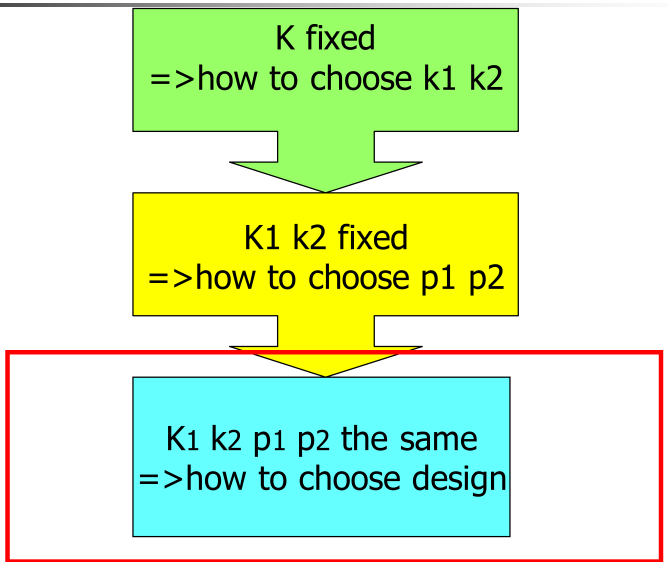
*D*2 : *I* = *ABpq* = *ACDpr* = *BCDqr* (4.3.0.2)
(q=Abp,r=ACDp)

TABLE 4. Catalog of Minimum Aberration 32-Run FFSP Designs (Designs are identified by $k_1.k_2.p_1.p_2$ and ordered by the number of factors $k = k_1 + k_2$.)

	$k_1.k_2.p_1.p_2$	WLP generators	WLP	# WP	(a)
	3.4.0.2	[illegible]	0 1 2	6	[illegible]
	3.4.1.1	ABC, Apqrs	1 0 1 1	4	0
D2	4.3.0.2	ABpq, ACDpr	0 1 2	10	2
D1	4.3.1.1	ABCD, ABpqr	0 1 2	8	0
	5.2.1.1	[illegible]	0 1 2	10	1
	5.2.1.1	[illegible]	0 1 2	10	1

Both designs have the same word length pattern, $W = (0, 1, 2, 0, 0)$.
*D*2 has $pq = AB$, $pr = ACD$, and $qr = BCD$.

*D*1 is better than *D*2 in its ability to detect significant 2fi's



Choosing Between Non-isomorphic MA design??

- Example 4: 32-runs, k1=2,k2=6,p1=0,p2=3

*D*1 : *I* =*Apqs*=*Aprt*=*ABqru*=*qrst*=*Bprsu*=*Bpqtu*=*ABstu*(2.6.0.3)

*D*2 : *I* =*ABpqs*=*ABprt*=*ABqru*=*qrst*=*prsu*=*pqtu*=*ABstu*(2.6.0.3)

*D*3 : *I* =*ABqs*=*Apqt*=*ABpru*=*Bpst*=*pqrsu*=*Bqrtu*=*Arstu*(2.6.0.3)

*D*4 : *I* =*ABqs*=*ABrt*=*Apqru*=*qrst*=*Bprsu*=*Bpqtu*=*Apstu*(2.6.0.3)

TABLE 4. Catalog of Minimum Aberration 32-Run FFSP Designs (Designs are identified by k_1, k_2, p_1, p_2 and ordered by the number of factors $k = k_1 + k_2$.)

	k_1, k_2, p_1, p_2	WLP generators	WLP	# WP	(a)
	1.7.0.3	Aprt, Aqru, Apqu	0 3 4	2	0
	1.7.0.3	pqrt, pqsu, Aprtu	0 3 4	2	0
D1	2.6.0.3	Apqs, Aprt, ABqru	0 3 4	4	0
D2	2.6.0.3	ABpqs, ABqrt, ABqru	0 3 4	4	0
D3	2.6.0.3	ABqs, Aqrt, ABpru	0 3 4	4	1
D4	2.6.0.3	ABqs, ABrt, Apqru	0 3 4	4	2
	3.5.0.3	ABCpr, ABCqs, Apqt	0 3 4	8	2

*D*1 and *D2* have no any SP 2fi's that are aliased with WP main effects or interactions involving only WP factors.
D3 has one SP 2fi aliased with a 2fi of only WP factors (i.e., *AB* = *qs*)
*D*4 has two SP 2fi's aliased with an interaction of only WP factors (i.e., *AB* = *qs* = *rt*).

*D*1 and *D*2 are equivalent, and better than *D*3 which is in turn better than *D*4.

Consider of Cost and Run-size

- In example 3: D1 & D2 design

TABLE 4. Catalog of Minimum Aberration 32-Run FFSP Designs (Designs are identified by k_1, k_2, p_1, p_2 and ordered by the number of factors $k = k_1 + k_2$.)

	k_1, k_2, p_1, p_2	WLP generators	WLP	# WP	(a)
	3.4.0.2	ABCpr, ABCqs	0 1 2	8	3
	3.4.1.1	ABC, Apqrs	1 0 1 1	4	0
D2	4.3.0.2	ABpq, ACDpr	0 1 2	16	3
D1	4.3.1.1	ABCD, ABpqr	0 1 2	8	0
	5.2.1.1	ABCE, ABDpq	0 1 2	16	1
	5.2.1.1	ABCDE, ABpq	0 1 2	16	1

D1 only 8 WP settings.D2 needs 16 WP settings

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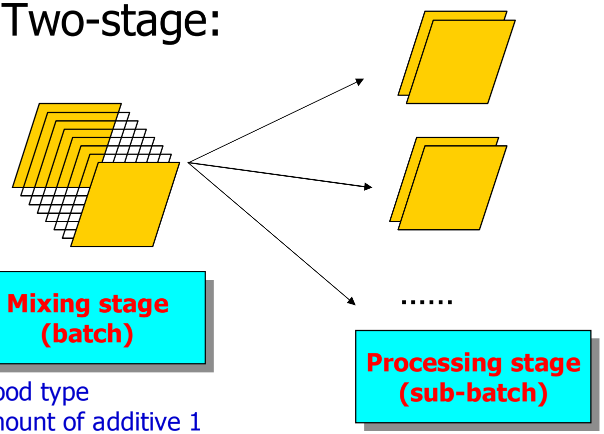
Wood Product Example

- Two-stage:



A - Wood type
B - Amount of additive 1
C - Amount of additive
D - Wood size
E - Wood moisture

p - Process time
q - Pressure
r - Material density.

Wood Product Example

- Prior knowledge:
  AB, AC, AD, AE, Ap, Aq, and *Ar ->negligible*
- Raw material expensive (i.e. A,B,C,D,E)
- In table 4

$2^{(5+3)-(2+1)}$: resolution III

$2^{(5+3)-(1+2)}$: resolution IV

- (5.3.1.2)

TABLE 4. Catalog of Minimum Aberration 32-Run FFSP Designs. (Designs are identified by $k_1.k_2.p_1.p_2$ and ordered by the number of factors $k = k_1 + k_2$.)

$k_1.k_2.p_1.p_2$	WLP generators	WLP	# WP	$\|\omega\|$
4.4.1.2	$ABCD$, $ABpq$, $ACpqr$	0 3 4	8	1
5.3.1.2	$ABCE$, $ABpq$, $ACDpr$	0 2 4	16	2
5.3.1.2	$ABCDE$, $ABpq$, $ACDpr$	0 2 4	16	2
5.3.2.1	ABD, ACE, $BCpqr$	2 1 2 2	8	0

$$D_1 : I = ABCE = ABpq = ACDpr = CEpq = BDEpr = BCDqr = ADEqr$$

(2)

$$D_2 : I = ABCDE = ABpq = ACDpr = CDEpq = BEpr = BCDqr = AEqr.$$

pq, pr, qr ->test against WP error term

- The possible 64-run FFSP designs are the $2^{(5+3)-(1+1)}$ and the $2^{(5+3)-(0+2)}$
  Using the algorithm of Bingham and Sitter (1999), we determine MA designs and find that the word length pattern for the MA design in both cases is W = (0, 0, 2, 1), and thus both of these designs have resolution V. Furthermore, the MA design is unique in both cases.
  The MA $2^{(5+3)-(1+1)}$ and $2^{(5+3)-(0+2)}$ FFSP designs are
  D3 : I = ABCDE = ABpqr = CDEpqr (5.3.1.1)
  D4 : I = ABCpq = CDEpr = ABDEqr. (5.3.0.2)

- D3 does not have any SP main effects or 2fi's aliased with WP main effects or interactions of only WP factors.
- D4 has *pq = ABC*, *pr = CDE*, and *qr = ABDE*
- conclude : *D3* is better than *D4*
- *Consider cost: D3* is better than *D4*

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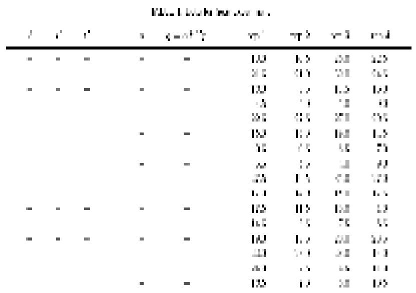
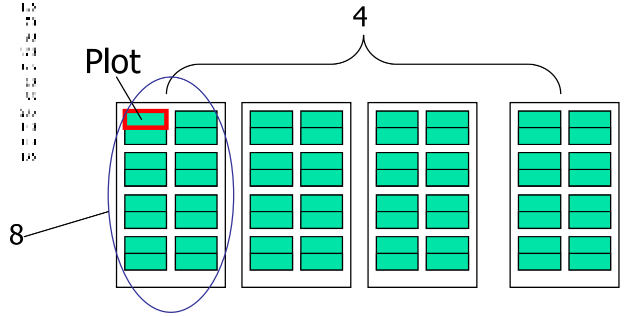
p: carbon potential *q:* operating mode



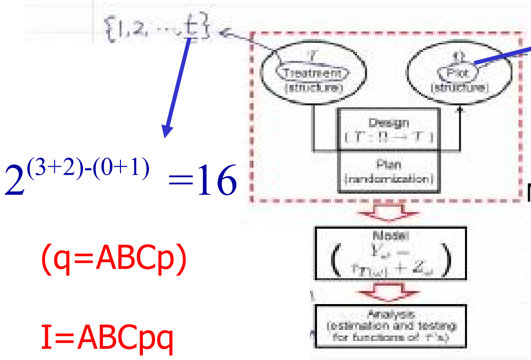
K1=3 K2=2 P1=0 P2=1

(q=ABCp) I=ABCpq

64 runs



16=t<N=64



$2^{(3+2)-(0+1)} = 16$

(q=ABCp)

I=ABCpq

N=64 plots, Nesting:

(4/8/2)

N(4,N(8,2))=1+(4-1)+4(N(8,2) -1)
=1+(4-1)+4($\cancel{1}$+(8-1)+8(2-1)-$\cancel{1}$)
=1+3+4(8-1)+4 8(2-1)
=1+3+28+32

S_0 S_1 S_2 S_3

- plan and the restrictions on randomization:

  -A,B,C to larger experimental unit.

  as in a complete-block design

  - p to small experimental unit.

  within each small block ,

  complete randomized design

Exp.Unit of p q

plot

Exp.Unit of A B C

Treatment structure	plot structure
A x B x C x p x q	U/V/W
2 2 2 2 2	4/8/2
	U1 U2 V1 V2 V3
	2 2 2 2 2

(q=ABCp)

I=ABCpq

Design key:

A	B	C	p	q
V1	V2	V3	W	V1V2V3W

Treatment strcture $\overline{\mathrm{V}}$ **Plot structure**

dim			dim
1	V_0	S_0	1
	$\oplus$	$\oplus$	
15	$\mathrm{W_T}$	S_1	3
	$\oplus$	$\oplus$	
48	$\mathrm{V_T^{\perp}}$	S_2	28
		$\oplus$	
		S_3	32

$\overline{\mathrm{V}}$

dim			dim
1	V_0	S_0	1
	$\oplus$	$\oplus$	
15	$\mathrm{W_T}$	S_1	3
	$\oplus$	$\oplus$	
48	$\mathrm{V_T^{\perp}}$	S_2	28
		$\oplus$	
		S_3	32

U1 ,U2,U1U2

V1,V2,V3,V1V2,V1V3,V2V3,V1V2V3
U1V1,U1V2,U1V3, U1V1V2,U1V1V3,U1V2V3,U1V1V2V3
U2V1,U2V2,U2V3, U2V1V2,U2V1V3,U2V2V3,U2V1V2V3
U1U2V1,U1U2V2,U1U2V3, U1U2V1V2,U1U2V1V3,U1U2V2V3,U1U2V1V2V3

W,WU1,WU2,WU1U2
W(U1V1,U1V2,U1V3, U1V1V2,U1V1V3,U1V2V3,U1V1V2V3)
W(U2V1,U2V2,U2V3, U2V1V2,U2V1V3,U2V2V3,U2V1V2V3)
W(U1U2V1,U1U2V2,U1U2V3, U1U2V1V2,U1U2V1V3,U1U2V2V3,U1U2V1V2V3)

Alias set

(I=ABCpq)

A=BCpq= V1
B=ACpq= V2
C=ABpq= V3
AB=Cpq= V1V2
AC=Bpq= V1V3
BC=Apq= V2V3
pq=ABC= V1V2V3
p=ABCq= W
q=ABCp= V1V2V3W
Ap=BCq= V1W
Aq=BCp= V2V3W
Bp=ACq= V2W
Bq=ACp= V1V3W
Cp=ABq= V3W
Cq=Abp= V1V2W

$\overline{\mathrm{V}}$

dim			dim
1	V_0	S_0	1
	$\oplus$	$\oplus$	
15	$\mathrm{W_T}$	S_1	3
	$\oplus$	$\oplus$	
48	$\mathrm{V_T^{\perp}}$	S_2	28
		$\oplus$	
		S_3	32

ANOVA table

Stratum	Source	d.f.	SS	EMS	VR
S0	mean	1	$\frac{sum^2}{64}$	$\lvert\tau_0\rvert^2+\xi_0$	-
S1	blocks	3	CSS(S1)-SS(S0)	ξ_1	-
S2	A, B, C, AB, AC, BC, pq	7	SS(.)	$\frac{1}{7}\lvert\tau_W\rvert^2-\xi_2$	$\frac{MS(W)}{MS(S2\ residual)}$
	residual	21	By subtraction	ξ_2	-
	total	28	CSS(S2)-CSS(S1)		
S3	p, q, Ap, Aq, Bp, Bq, Cp, Cq	8	SS(.)	$\frac{1}{8}\lvert\tau_S\rvert^2+\xi_3$	$\frac{MS(S)}{MS(S3\ residual)}$
	residual	24	By subtraction	ξ_3	-
	total	32	$\sum y_\omega^2-CSS(S2)$		

Thanks of your attention

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