多階層實驗設計期末報告

L^k Fractional Experiment With Hard-To Change and Easy-To-Change Factors

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What is hard-to-change factor

改變因子狀需要耗掉許多時間或耗大比經費不符合經濟成本

此報告中 X_1 代表hard-to-change 因子

例如：改變鍋爐狀態 或者 紙漿的問題

Five different run order scenarios

- Completely randomized
- RRO
- Completely restricted
- Partially restricted
- Classical split-plot procedure

Completely randomized

完全的隨機排列

例如

二因子二水平的設計共有四階層的排列方式

RRO

Random run order with the hard-to-change factor is not reset successive runs have the same level of that factor

例子

X_1 is reset at runs 1,3,6,8

TABLE 1. A Run Order for a 2^3 Experiment

run	X_1	X_2	X_3	Block
1	−	+	−	1
2	−	−	+	1
3				2
4	−	+	+	2
5		|		2
6				3
7	−	+	+	3
8	−	−	+	4

Completely restricted

Hard-to-change factor各個level出現在相同的block之中

Partially restricted

The runs at each level of the hard-to-change factor are randomly divided into two groups of equal size or of sizes that differ by a run

Covariance Matrix(V) for the Run Order Scenarios

ε_S be the error of easy-to-change factor With a mean zero and a various σ_S^2

ε_W be the error of hard-to-change factor With a mean zero and a various σ_W^2

Covariance Matrix(V) for the Run Order Scenarios-續

$$V = (\sigma_S^2 + (1-p)\sigma_W^2)I + (p\sigma_W^2)U_r U_r^t$$

p is the probability that any two runs having the same setting of the hard-to-change factor are in the same block

Covariance Matrix(V) for the Run Order Scenarios-續

Completely restricted p=1

Completely randomized p=0

Partially restricted

--block is even p= $\dfrac{L^{k-1}-2}{2(L^{k-1}-1)}$

--block is odd p= $\dfrac{L^{k-1}-1}{2(L^{k-1})}$

RRO p= $\dfrac{2}{L^{k-1}(L-1)+2}$

- For completely randomized

  $\mathbf{V}_{cr} = (\sigma_S^2 + \sigma_W^2)\mathbf{I} = \sigma^2\mathbf{I}$

  because the whole-plot factor reset on each run
- For a completely restricted

  $\mathbf{V}_r = \sigma_S^2\mathbf{I} + \sigma_W^2\mathbf{U}_r\mathbf{U}_r'$

  $\mathbf{U}_r = \begin{bmatrix} \mathbf{J}_1 & & \\ & \vdots & \\ & & \mathbf{J}_L \end{bmatrix}$ $\quad\mathbf{J}_i = [1, 1 \cdots 1]_{L^{k-1}}.$

For a RRO design

the 2^3 experiment：3,1,4,5,8,7,6,2

$$\mathbf{X} = \begin{array}{c} \text{run} \\ 3 \\ 1 \\ 4 \\ 5 \\ 8 \\ 7 \\ 6 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} X_1 & X_2 & X_3 \end{array} \\ \begin{pmatrix} - & + & - \\ - & - & - \\ - & + & - \\ + & - & - \\ + & + & \\ + & + & \\ + & - & - \\ - & - & - \end{pmatrix} \end{array} \qquad \mathbf{U} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$\mathbf{X} = \begin{array}{c} \text{run} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array} \begin{array}{c} \begin{array}{ccc} X_1 & X_2 & X_3 \end{array} \\ \begin{pmatrix} - & - & - \\ & & + \\ - & - & - \\ & & + \\ + & - & - \\ + & & + \\ + & - & - \\ + & & + \end{pmatrix} \end{array} \qquad \mathbf{U} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}. \qquad \mathbf{U}\mathbf{U}' = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

V(b_i) in the L^k system

$Y = X\beta + \varepsilon$

For L^k experiment using different run order scenarios

$$
\begin{aligned}
Cov(\mathbf{b}) &= (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{V}\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1} \\
&= \sigma_S^2(\mathbf{X}'\mathbf{X})^{-1} \\
&\quad + \sigma_W^2(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\left[(1-p)\mathbf{I} + p\mathbf{U}\mathbf{U}'\right]\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1} \\
&= \sigma_S^2 f_1(\mathbf{X}) + \sigma_W^2 f_2(\mathbf{X}).
\end{aligned}
$$

Compare the variance of b_i

$$V(b_i) = a_i\sigma_S^2 + c_i\sigma_W^2 \qquad i = 0, 1, \ldots,$$

a and c value for 2^k experiment with X_1 Hard-To-Change

Factor	All Randomizations	General Formula	Completely Restricted	Partially Restricted	Random Run Order	Completely Randomized	Orthogonal Blocking
	a	c	c	c	c	c	c
$i = 1$	$\frac{1}{2^k}$	$\frac{(1-p)}{2^k} + \frac{p}{2}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{2}{4(2^{k-2}+1)}$	$\frac{1}{2^k}$	$\frac{4}{2^k}$
$i \neq 1$	$\frac{1}{2^k}$	$\frac{(1-p)}{2^k}$	0	$\frac{1}{4(2^{k-1}-1)}$	$\frac{1}{4(2^{k-2}+1)}$	$\frac{1}{2^k}$	0

Example: 2^4 System With X_1 Hard-To-Change

Run Order Scenario	$V(b_1)$	$V(b_2) = V(b_3) = \ldots$
Completely Restricted	$\frac{1}{2}\sigma_W^2 + \frac{1}{16}\sigma_S^2$	$\frac{1}{16}\sigma_S^2$
Partially Restricted	$\frac{1}{4}\sigma_W^2 + \frac{1}{16}\sigma_S^2$	$\frac{1}{28}\sigma_W^2 + \frac{1}{16}\sigma_S^2$
Random Run Order	$\frac{2}{20}\sigma_W^2 + \frac{1}{16}\sigma_S^2$	$\frac{1}{20}\sigma_W^2 + \frac{1}{16}\sigma_S^2$
Completely Randomized	$\frac{1}{16}\sigma_W^2 + \frac{1}{16}\sigma_S^2$	$\frac{1}{16}\sigma_W^2 + \frac{1}{16}\sigma_S^2$
Orthogonal Blocking	$\frac{4}{16}\sigma_W^2 + \frac{1}{16}\sigma_S^2$	$\frac{1}{16}\sigma_S^2$

Completely randomized v.s. completely restricted

- In hard-to-change factor
  completely randomized is better
- In easy-to-change factor
  completely restricted is better
- About price
  completely restricted is better

Split plot v.s. partially restricted

- For the hard-to-change factor
  both has the same standard error
  (because they have four blocks)
- For all other factors
  split plot is better than partially restricted

Classical Split Plot Blocking

Two different scheme

1. four blocks $I = X_1 = X_1X_2X_3X_4$
2. eight blocks $I = X_1 = X_2X_3 = X_1X_2X_4$

For the four blocks scheme

X matrix is written in the standard order

$$\mathbf{U} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

2^4 G-efficiency

When costs and prediction variance are criteria, G-efficiency is important

Completely randomized 2^4 factorial is a G-optimum experiment

$11(\sigma_S^2 + \sigma_W^2)/16$ for a main effects plus interaction model

G-efficiency--(continued)

- Split –plot experiment with four blocks the maximum variance is $\sigma_S^2/2 + 11\sigma_W^2/16$
- Split-plot design can be superior to “optimum” designs

Concluding Remarks

- When errors are introduced by changing the level of the factors,

  RRO and completely randomized are not well for easy-to-change factor
- Restricted experiments are more economical

  but there is a large variance for the hard-to-change factor

Concluding Remarks--(continued)

- Some people use partially restricted

  it is not good

  we can use Split plot blocking procedures