Analyzing Unreplicated Blocked or Split-Plot Fractional Factorial Designs

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- Blocked Fractional Factorial Designs
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Fractional Factorial Designs

- In many industrial settings, experimental replication is sacrificed for run size. This can present serious difficulties in the analysis.
- An experimenter wishes to test various factors. In order to test many factors in an experiment, many industrial experiments are performed using a fractional factorial(FF) design.
- FF designs are run by assigning additional factors to the unused runs in a full factorial design with k-p factors.

Blocked Fractional Factorial Designs

- If we wish to run an experiment in 2^p blocks, we can use multiple effect columns and suitable replacement rules.
- For example, to run an experiment in four blocks, we can use

b_1	b_2	b_1b_2	Block Indicator
-1	-1	$+1$	0
$+1$	-1	-1	1
-1	$+1$	-1	2
$+1$	$+1$	$+1$	3 .

Blocked Fractional Factorial Designs(cont.)

- The model for a blocked factorial design may be summarized by

$$y = f(\text{Treatments}) + \varepsilon + f(\text{Blocks}) + e,$$

where ε and e are the treatment and block error terms, and $f(\cdot)$ and $g(\cdot)$ are functions of the treatments and block effects. It is also assumed that ε and e are mutually independent, such that $\varepsilon \sim N(0,\sigma^2_{Trt})$ and $e \sim N(0,\sigma^2_B)$.

- When an experimenter wishes to run a FF design and restrictions result in the need for blocking, the alias structure is slightly more complicated.

Blocked Fractional Factorial Designs (cont.)

- **Example1：**

A blocked FF design was performed where the fractionation for the treatments was based on E = ABC and F = BCD, and the blocks were chosen by assigning b_1 = ACD and b_2 = ABD. This implies that I = ABCE = BCDF = $ACDb_1$ = $ABDb_2$, with the defining contrast subgroup

I = ABCE = BCDF = $ACDb_1$ = $ABDb_2$ = ADEF = $BDEb_1$ = $CDEb_2$ = $ABFb_1$ = $ACFb_2$ = BCb_1b_2 = $CEFb_1$ = $BEFb_2$ = AEb_1b_2 = DFb_1b_2 = $ABCDEFb_1b_2$

Blocked Fractional Factorial Designs (cont.)

We denoted a blocked FF by $2^{(k+m)-(m+p)}$, where k is the number of factors under consideration, p is the degree of fractionation for the k factors, and m is the number of blocking factors.

In the above experiment, we have $2^{(6+2)-(2+2)}$ blocked FF. A particular alias string would be found in the same fashion as outlined previously; however, any interaction between blocks and factors is assumed to be negligible.

Methods for Analysis of Blocked FF Designs

➢ **Half-Normal Plots**

- Plotting $\Phi^{-1}(0.5+0.5[i+0.5]/(n-1))$ against $|\hat{\beta}|_{(i)}$ for i=1,2,…n, where we have the ordered absolute effects $|\hat{\beta}|_{(1)} \le |\hat{\beta}|_{(2)} \le \cdots |\hat{\beta}|_{(n)}$ and $\Phi(\cdot)$ is the cumulative distribution function of a standard normal.
- In order to use a half-normal probability plot, all of the estimators must have the same variance.
- The factorial effects all have the same variance and are independent of blocks, but the estimated block effects have a different variance.
- This involves a plot of $\Phi^{-1}(0.5+0.5[i+0.5]/r)$ against $|\hat{\beta}|_{(i)}$ i=1,2,…,r, r=n-2^m+1 factorial effects $\beta_1,\cdots,\beta_r$.

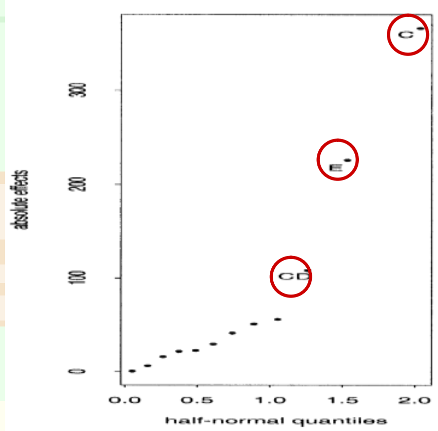
Methods for Analysis of Blocked FF Designs (cont.)

- Consider example 1 ,the data from the experiment.

std. order	block	b_1	b_2	b_1b_2	Design Factors A	B	C	D	E	F	Mean
1	I	−1	−1	1	−1	−1	−1	−1	−1	−1	1085
8	I	−1	−1	1	1	1	1	−1	1	−1	1357
10	I	−1	−1	1	1	−1	−1	1	1	1	377
15	I	−1	−1	1	−1	1	1	1	−1	1	1910
3	II	−1	1	−1	−1	1	−1	−1	1	1	697
6	II	−1	1	−1	1	−1	1	−1	−1	1	1738
12	II	−1	1	−1	1	1	−1	1	−1	−1	959
13	II	−1	1	−1	−1	−1	1	1	1	−1	1274
4	III	1	−1	−1	1	1	−1	−1	−1	1	1261
5	III	1	−1	−1	−1	−1	1	−1	1	1	1118
11	III	1	−1	−1	−1	1	−1	1	1	−1	516
14	III	1	−1	−1	1	−1	1	1	−1	−1	1784
2	IV	1	1	1	1	−1	−1	−1	1	−1	782
7	IV	1	1	1	−1	1	1	−1	−1	−1	1675
9	IV	1	1	1	−1	−1	−1	1	−1	1	702
16	IV	1	1	1	1	1	1	1	1	1	1378

Methods for Analysis of Blocked FF Designs (cont.)

- $n=2^4-1=15$, $m=2$, $r=15-2^2+1=12$
- From the plot we can see that there were two or three possible significant effects.



Methods for Analysis of Blocked FF Designs (cont.)

- **Lenth's Method**
  - Pseudo standard error (PSE) to estimate the standard deviation of the n factorial effect estimates, $\hat{\beta}_i$.
  - We consider the factorial effect estimates $\hat{\beta}_1,\cdots,\hat{\beta}_r$ free of the blocking effects and the mean.
  - Define : $PSE = 1.5 \cdot \text{median}_{\{|\hat{\beta}_i|<2.5s_0,\, i=1,\ldots,r\}} |\hat{\beta}_i|$

    where $s_0 = 1.5 \cdot \text{median}_{\{i=1,\cdots,r\}} |\hat{\beta}_i|$. The test statisric is then calculated by dividing the effect estimates by the PSE, i.e. ,

$$t_{Lenth,i} = \frac{\hat{\beta}_i}{PSE}, \quad \text{for } i = 1,\cdots,r.$$

Methods for Analysis of Blocked FF Designs (cont.)

- Two types of error,

  Individual Error Rate (IER) :

  the average proportion of inactive effects declared active.

  Experiment-wise Error Rate (EER) :

  the proportion of models which are incorrectly identified.

# runs	# blocks	# effects	α 0.005	0.01	0.05	0.1	0.2	0.4
					IER			
8	1	7	16.804	5.102	2.329	1.732	1.207	0.748
	2	6	7.429	5.467	2.211	1.645	1.157	0.775
	4	4	9.012	6.363	2.033	1.483	1.069	0.769
16	1	15	4.489	3.678	2.162	1.705	1.257	0.793
	2	14	4.576	3.774	2.161	1.698	1.250	0.796
	4	12	4.960	4.023	2.196	1.705	1.241	0.790
	8	8	6.134	4.746	2.224	1.673	1.197	0.780
32	1	31	3.462	3.048	2.060	1.678	1.276	0.817
	2	30	3.467	3.070	2.063	1.680	1.275	0.817
	4	28	3.512	3.098	2.067	1.682	1.272	0.816
	8	24	3.690	3.207	2.087	1.689	1.270	0.812
	16	16	4.241	3.535	2.133	1.691	1.258	0.799
64	1	63	3.128	2.801	2.015	1.669	1.281	0.831
	2	62	3.129	2.804	2.018	1.670	1.282	0.831
	4	60	3.142	2.809	2.019	1.670	1.282	0.830
	8	56	3.169	2.831	2.025	1.671	1.280	0.829
	16	48	3.234	2.880	2.034	1.675	1.281	0.827
	32	32	3.494	3.047	2.064	1.683	1.273	0.819

# runs	# blocks	# effects	α 0.005	0.01	0.05	0.1	0.2	0.4
					EER			
8	1	7	13.383	9.417	4.920	3.704	2.452	1.818
	2	6	13.457	10.820	5.240	3.717	2.256	1.692
	4	4	17.835	12.663	5.313	2.964	1.922	1.412
16	1	15	7.900	6.756	4.265	3.522	2.850	2.152
	2	14	8.064	6.726	4.407	3.583	2.821	2.109
	4	12	8.549	7.090	4.547	3.619	2.767	2.053
	8	8	10.689	8.475	4.895	3.674	2.400	1.838
32	1	31	5.613	5.100	3.940	3.433	2.979	2.412
	2	30	5.575	5.073	3.933	3.438	2.995	2.403
	4	28	5.711	5.207	3.946	3.449	2.984	2.377
	8	24	6.214	5.460	4.014	3.475	2.949	2.316
	16	16	7.477	6.218	4.226	3.508	2.839	2.155
64	1	63	4.906	4.567	3.799	3.462	3.128	2.714
	2	62	4.915	4.559	3.806	3.470	3.124	2.712
	4	60	4.962	4.552	3.817	3.464	3.122	2.697
	8	56	5.002	4.598	3.838	3.466	3.097	2.677
	16	48	5.091	4.694	3.823	3.458	3.065	2.634
	32	32	5.919	5.282	3.943	3.448	2.993	2.427

Methods for Analysis of Blocked FF Designs (cont.)

Re-analyzed example 1:

- The estimated effects C = 365.94, E = −225.94, CD = 108.06, B = 55.81, D = −50.81, A = 41.19, AD = −29.19, BD = 22.44, AB = −21.56, F = −15.69, AC = −6.19, and AF = −0.31
- s_0 = 52.781, PSE = 33 ,
- Effects are declared significant if $|t_{Lenth,i}| = |\hat{\beta}_i / PSE|$ is greater than the critical value.

  (5% IER = 2.196 , EER = 4.547)
- IER: C, E, CD as significant effects.
- EER: C, E as significant effects. → more conservative

Fractional Factorial Split-Plot Designs

- In many industrial experiments, even when there is no blocking, it is not possible to perform all of the runs in a completely randomized order. This restriction on randomization will result in a split-plot structure.
- The design will be denoted as a $2^{(k_1+k_2)-(p_1+p_2)}$ FFSP, where k_1 is the number of WP (whole-plot) factors, p_1 is the degree of fractionation at the whole plot level, k_2 is the number of SP (sub-plot) factors, and p_2 is the degree of fractionation for the sub-plot design.

Fractional Factorial Split-Plot Designs(cont.)

- The model for a full factorial split-plot can be summarized as $y = f$(WP effects) + ε + g(SP effects) + e , where ε and e are the WP and SP error terms, and $f(\cdot)$ and $g(\cdot)$ are functions of the WP and SP terms.
- ε and e are mutually indenpdent, such that $\varepsilon \sim N(0,\sigma^2_{WP})$ and $e \sim N(0,\sigma^2_{SP})$.
- The two different error terms imply that not all effects have the same variance.

Methods for Analysis of FFSP Designs

- **Example 2:**

  k_1=8,(WP factors = A, B, C, D, E, F, G, H)

  k_2=3,(SP factors = p, q, r)

  p_1=4,($E = ABD, F = ABC, G = BCD, and\ H = ACD$)

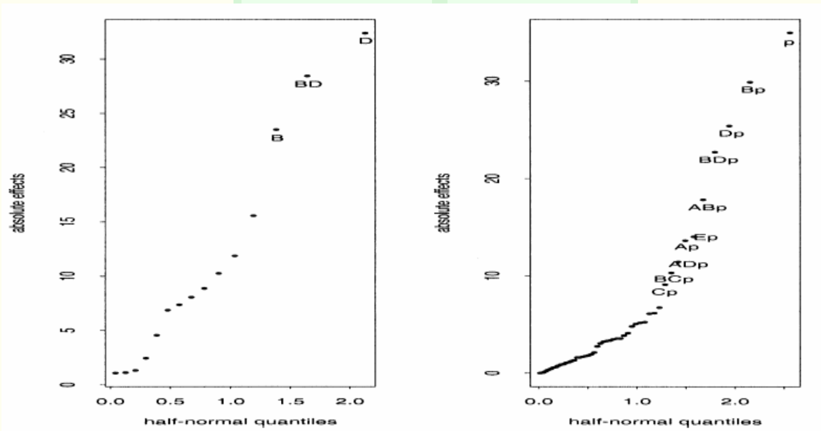
  p_2=1,($r = pq$)

Methods for Analysis of FFSP Designs (cont.)

➢ **Half-Normal Plot**

- Stratum: $S_0 \oplus S_1 \oplus S_2$,

  S_1 and S_2 have different variance



Methods for Analysis of FFSP Designs(cont.)

➢ **Lenth's Method**

- Since the test is applied as a randomized designs for the WP factors, one should use the critical values.
- The SP effects are tested as a blocked FF design, and thus the critical values presented in Table.

From example 2:

- Applying Lenth's method to the WP effects again declared D, BD, and B as significant.
- The test on the SP effects declared p, Bp, DP, BDp, ABp = DEp, Ep, Ap, ADp, and BCp.

Example

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A unreplicated FFSP Designs

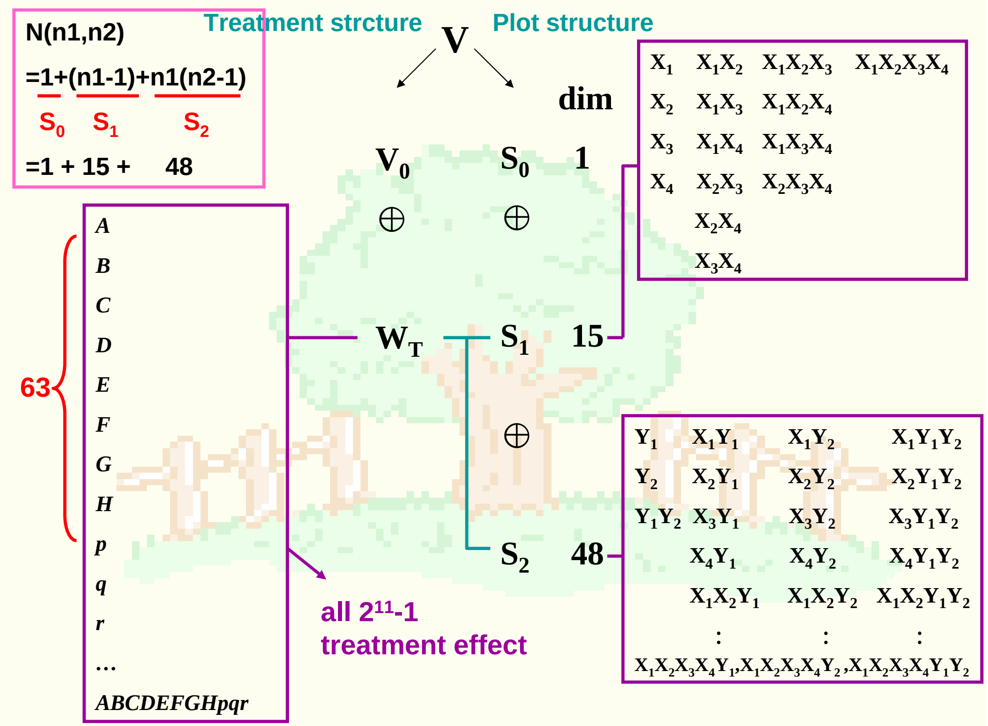
Treatment structure: $A \times B \times C \times D \times E \times F \times G \times H \times p \times q \times r$ (2 2 2 2 2 2 2 2 2 2 2)

plot structure: $X \;/\; Y$ (16, 4)

$X_1\ X_2\ X_3\ X_4\ \ Y_1\ Y_2$ (2 2 2 2 2 2)

Design Key

A	B	C	D	E	F	G	H	p	q	r
X1	X2	X3	X4	X1X2X4	X1X2X3	X2X3X4	X1X3X4	Y1	Y2	Y1Y2



Alias structure

I = ABDE = ABCF = BCDG = ACDH = pqr = CDEF = ACEG = BCEH = ABDEpqr = ADFG
= BDFH = ABCFpqr = ABGH = BCDGpqr = ACDHpqr =BEFG = AEFH = CDEFpqr = DEGH
= ACEGpqr = BCEHpqr = CFGH = ADFGpqr = BDFHpqr = ABGHpqr = ABCDEFGH
= BEFGpqr = AEFHpqr = DEGHpqr = FGHpqr = ABCEFGHpqr

Coufounded

A = BDE = BCF = ABCDG = CDH = Apqr = ACDEF = CEG = ABCEH = BDEpqr = DFG
= ABDFH = BCFpqr = BGH = ABCDGpqr = CDHpqr = ABEFG = EFH = ACDEFpqr
= ADEGH = CEGpqr = ABCEHpqr = ACFGH = DFGpqr = ABDFHpqr = BGHpqr
=BCDEFGH = ABEFGpqr = EFHpqr = ADEGHpqr = AFGHpqr = BCEFGHpqr = X1

B = ADE = ACF = CDG = ABCDH = Bpqr = BCDEF = ABCEG = CEH = ADEpqr = ABDFG
= DFH = ACFpqr = AGH = CDGpqr = ABCDHpqr =EFG = ABEFH = BCDEFpqr = BDEGH
= ABCEGpqr = CEHpqr = BCFGH = ABDFGpqr = DFHpqr = AGHpqr = ACDEFGH
= EFGpqr = ABEFHpqr = BDEGHpqr = BFGHpqr = ACEFGHpqr = X2

C = ABCDE = ABF = BDG = ADH = Cpqr = DEF = AEG = BEH = ABCDEpqr = ACDFG
= BCDFH = ABFpqr = ABCGH = BDGpqr = ADHpqr =BCEFG = ACEFH = DEFpqr
= CDEGH = AEGpqr = BEHpqr = FGH = ACDFGpqr = BCDFHpqr = ABCGHpqr
= ABDEFGH = BCEFGpqr = ACEFHpqr = CDEGHpqr = CFGHpqr = ABEFGHpqr
= X3

D = ABE = ABCDF = BCG = ACH = Dpqr = CEF = ACDEG = BCDEH = ABEpqr = AFG
= BFH = ABCDFpqr = ABDGH = BCGpqr = ACHpqr =BDEFG = ADEFH = CEFpqr
= EGH = ACDEGpqr = BCDEHpqr = CDFGH = AFGpqr = BFHpqr = ABDGHpqr
= ABCEFGH = BDEFGpqr = ADEFHpqr = EGHpqr = DFGHpqr = ABCDEFGHpqr
= X4

E = ABD = ABCEF = BCDEG = ACDEH = Epqr = CDF = ACG = BCH = ABDpqr
= ADEFG = BDEFH = ABCEFpqr = ABEGH = BCDEGpqr = ACDEHpqr =BFG = AFH
= CDFpqr = DGH = ACGpqr = BCHpqr = CEFGH = ADEFGpqr = BDEFHpqr
= ABEGHpqr = ABCDFGH = BFGpqr = AFHpqr = DGHpqr = EFGHpqr
= ABCFGHpqr = X1X2X4

F = ABDEF = ABC = BCDFG = ACDFH = Fpqr = CDE = ACEFG = BCEFH = ABDEFpqr
= ADG = BDH = ABCpqr = ABFGH = BCDFGpqr = ACDFHpqr =BEG = AEH =CDEpqr
= DEFGH = ACEFGpqr = BCEFHpqr = CGH = ADGpqr = BDHpqr = ABFGHpqr
= ABCDEGH = BEGpqr = AEHpqr = DEFGHpqr = GHpqr = ABCEGHpqr = X1X2X3

G = ABDEG = ABCFG = BCD = ACDGH = Gpqr = CDEFG = ACE = BCEGH = ABDEGpqr
= ADF = BDFGH = ABCFGpqr = ABH = BCDpqr = ACDGHpqr =BEF = AEFGH
=CDEFGpqr = DEH = ACEpqr = BCEGHpqr = CFH = ADFpqr = BDFGHpqr
= ABHpqr = ABCDEFH = BEFpqr = AEFGHpqr = DEHpqr = FHpqr = ABCEFHpqr
= X2X3X4

H = ABDEH = ABCFH = BCDGH = ACD = Hpqr = CDEFH = ACEGH = BCE = ABDEHpqr
= ADFGH = BDF = ABCFHpqr = ABG = BCDGHpqr = ACDpqr =BEFGH = AEF
= CDEFHpqr = DEG = ACEGHpqr = BCEpqr = CFG = ADFGHpqr = BDFpqr
= ABGpqr = ABCDEFG = BEFGHpqr = AEFpqr = DEGpqr = FGpqr = ABCEFGpqr
= X1X3X4

p = ABDEp = ABCFp = BCDGp = ACDHp = qr = CDEFp = ACEGp = BCEHp
=ABDEqr = ADFGp = BDFHp = ABCFqr = ABGHp = BCDGqr = ACDHqr =BEFGp
= AEFHp = CDEFqr = DEGHp = ACEGqr = BCEHqr = CFGHp = ADFGqr = BDFHqr
= ABGHqr = ABCDEFGHp = BEFGqr = AEFHqr = DEGHqr = FGHqr
= ABCEFGHqr = Y1

q = ABDEq = ABCFq = BCDGq = ACDHq = pr = CDEFq = ACEGq = BCEHq
=ABDEpr = ADFGq = BDFHq = ABCFpr = ABGHq = BCDGpr = ACDHpr =BEFGq
= AEFHq = CDEFpr = DEGHq = ACEGpr = BCEHpr = CFGHq = ADFGpr = BDFHpr
= ABGHpr = ABCDEFGHq = BEFGpr = AEFHpr = DEGHpr = FGHpr
= ABCEFGHpr = Y2

r = ABDEr = ABCFr = BCDGr = ACDHr = pq = CDEFr = ACEGr = BCEHr = ABDEpq
= ADFGr = BDFHr = ABCFpq = ABGHr = BCDGpq = ACDHpq =BEFGr = AEFHr
= CDEFpq = DEGHr = ACEGpq = BCEHpq = CFGHr = ADFGpq = BDFHpq
= ABGHpq = ABCDEFGHr = BEFGpq = AEFHpq = DEGHpq = FGHpq
= ABCEFGHpq = Y1Y2

AB = ... = X1X2	Dr = ... = X4Y1Y2	ACr = ... = X1X3Y1Y2
AC = ... = X1X3	Ep = ... = X1X2X4Y1	ADp = ... = X1X4Y1
AD = ... = X1X4	Eq = ... = X1X2X4Y2	ADq = ... = X1X4Y2
AE = ... = X2X4	Er = ... = X1XX2X4Y1Y2	ADr = ... = X1X4Y1Y2
AF = ... = X2X3	Fp = ... = X1X2X3Y1	AEp = ... = X2X4Y1
AG = ... = X1X2X3X4	Fq = ... = X1X2X3Y2	AEq = ... = X2X4Y2
AH = ... = X3X4	Fr = ... = X1X2X3Y1Y2	AEr = ... = X2X4Y1Y2
Ap = ... = X1Y1	Gp = ... = X2X3X4Y1	AFp = ... = X2X3Y1
Aq = ... = X1Y2	Gq = ... = X2X3X4Y2	AFq = ... = X2X3Y2
Ar = ... = X1Y1Y2	Gr = ... = X2X3X4Y1Y2	AFr = ... = X2X3Y1Y2
Bp = ... = X2Y1	Hp = ... = X1X3X4Y1	AGp = ... = X1X2X3X4Y1
Bq = ... = X2Y2	Hq = ... = X1X3X4Y2	AGq = ... = X1X2X3X4Y2
Br = ... = X2Y1Y2	Hr = ... =X1X3X4Y1Y2	AGr = ... = X1X2X3X4Y1Y2
Cp = ... = X3Y1	ABp = ... = X1X2Y1	AHp = ... = X3X4Y1
Cq = ... = X3Y2	ABq = ... = X1X2Y2	AHq = ... = X3X4Y2
Cr = ... = X3Y1Y2	ABr = ... = X1X2Y1Y2	AHr = ... = X3X4Y1Y2
Dp = ... = X4Y1	ACp= ... = X1X3Y1	
Dq = ... = X4Y2	ACq = ... = X1X3Y2	

- restrictions on the allocation of treatments to plots :
  - Treatment main factors can confounded with some of sub-plot effects, but cannot confounded with whole-plot effects.
- plan and the restrictions on randomization:
  - Apply treatment A,B,…,H to larger expimential unit.
  - Apply trearment p,q,r to each whole-plot as completely randomized design.

ANOVA table

stratum	Source	d.f.	S.S.	M.S.
S_0	mean	1	$\frac{\text{sum}^2}{64}$	$\lVert\tau_0\rVert^2+\xi_0$
S_1	**A, B, C, D, E, F, G, H, AB, AC, AD, AE, AF, AG, AH**	15	$\lVert P_1 y\rVert^2$	ξ_1
S_2	p, q, r	3	$SS(\cdot)$	$\frac{1}{3}\lVert\tau\rVert^2+\xi_2$
	Residual	45	$\lVert y-Gy\rVert^2-SS(\cdot)$	ξ_2
Total		64	$\lVert y-Gy\rVert^2$	