# 24 Run Split-Plot Experiments For Robust Parameter Design

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## The goal of the experiment

- Main effects for the whole plot factors
- Main effects for the subplot factors
- Two-factor interactions between the whole plot and subplot factors
- Two-factor interactions among subplot factors

## Outline

- Constructing 24 Run Designs
  Use confounding
  Based on a block design
- Comparing 24-run and 32-run designs
- An example
- Analysis

## Yates notation

- If a factor is at its low level, the corresponding letter is omitted from the treatment designation. Conversely, if a factor is at its high level, the corresponding letter is included.
- When all of the factors are at their low level, the treatment is designated by the symbol (1).

# Semifold

- Method:

  Use half of a foldover design

- Example: I=ABC

TABLE 1. Example of Semifolding

| | Original Points A | B | C | D | Treatment |
|---|---|---|---|---|---|
| 1 | − | − | − | − | (1) |
| 2 | − | − | + | + | cd |
| 3 | − | + | − | + | bd |
| 4 | − | + | + | − | bc |
| 5 | + | − | − | + | ad |
| 6 | + | − | + | − | ac |
| 7 | + | + | − | − | ab |
| 8 | + | + | + | + | abcd |
| | **Additional Points A** | **B** | **C** | **D** | **Treatment** |
| 9 | + | − | − | − | a |
| 10 | + | − | + | + | acd |
| 11 | + | + | − | + | abd |
| 12 | + | + | + | − | abc |

# MA designs for 16 runs

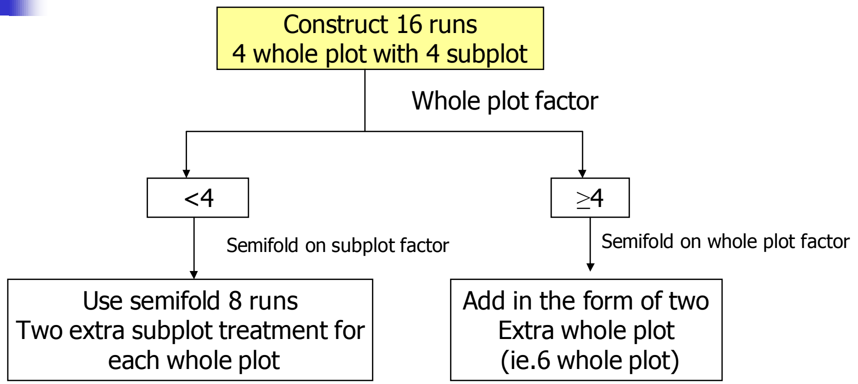
- MA designs use eight whole plots with two subplots per whole plot

  1.cost consideration

  2. It is not an efficient allocation of the degrees of freedom (df).

  3. it often results in a less sensitive test of treatment differences.

# Constructing 24 Run Designs using confounding




# Case1:2WP factors (A,B) and 4 SP factors (P,Q,R,S)

- I=PQR=QRS=PS

TABLE 2. 24 Point Design With of 2 WP and 4 SP Factors Using the Same Fraction [HP(HS) denotes high $P(S)$ and LP(LS) denotes low $P(S)$]

| $a$ | $b$ | $ab$ | $(1)$ |
|---|---|---|---|
| $q$ | $q$ | $q$ | $q$ |
| $r$ | $r$ | $r$ | $r$ |
| $ps$ | $ps$ | $ps$ | $ps$ |
| $pqrs$ | $pqrs$ | $pqrs$ | $pqrs$ |
| **Fold on HP** | **LP** | **HS** | **LS** |
| $s$ | $pq$ | $p$ | $qs$ |
| $qrs$ | $pr$ | $pqr$ | $rs$ |

# Case1:2WP factors (A,B) and 4 SP factors (P,Q,R,S)

- I=APQR=BQRS=ABPS

TABLE 3. 24 Point Design With of 2 WP and 4 SP Factors Using Split-Plot Confounding [HP(HS) denotes high $P(S)$ and LP(LS) denotes low $P(S)$]

| $a$ | $b$ | $ab$ | $(1)$ |
|---|---|---|---|
| $p$ | $s$ | $q$ | $qr$ |
| $pqr$ | $pq$ | $r$ | $pqs$ |
| $qs$ | $pr$ | $ps$ | $prs$ |
| $rs$ | $qrs$ | $pqrs$ | $(1)$ |
| | Fold on | | |
| HP | LP | HS | LS |
| $(1)$ | $ps$ | $p$ | $qrs$ |
| $qr$ | $pqrs$ | $pqr$ | $s$ |

# Case2:3WP factors (A,B,C) and 3 SP factors (P,Q,R,S)

- I = ABC =PQR = ABCPQR

TABLE 4. 24 Point Design With of 3 WP and 3 SP Factors Using the Same Fraction [HP denotes high $P$ and LP denotes low $P$]

| $a$ | $b$ | $c$ | $abc$ |
|---|---|---|---|
| $p$ | $p$ | $p$ | $p$ |
| $q$ | $q$ | $q$ | $q$ |
| $r$ | $r$ | $r$ | $r$ |
| $pqr$ | $pqr$ | $pqr$ | $pqr$ |
| | Fold on | | |
| HP | LP | LP | HP |
| $(1)$ | $pq$ | $pq$ | $(1)$ |
| $qr$ | $pr$ | $pr$ | $qr$ |

# Case2:3WP factors (A,B,C) and 3 SP factors (P,Q,R,S)

- I = ABC = ABPQR = CPQR

TABLE 5. 24 Point Design With of 3 WP and 3 SP Factors Using Split-Plot Confounding [HP denotes high $P$ and LP denotes low $P$]

| $a$ | $b$ | $c$ | $abc$ |
|---|---|---|---|
| $pq$ | $pq$ | $p$ | $p$ |
| $pr$ | $pr$ | $q$ | $q$ |
| $qr$ | $qr$ | $r$ | $r$ |
| $(1)$ | $(1)$ | $pqr$ | $pqr$ |
| | Fold on | | |
| HP | LP | LP | HP |
| $q$ | $pqr$ | $pq$ | $(1)$ |
| $r$ | $p$ | $pr$ | $qr$ |

# Case 3: 4 WP Factors (A, B, C, D) and 3 SP Factors (P, Q, R)

- I = ABC =BCD = AD = PQR = ABCPQR = BCDPQR =ADPQR

TABLE 6. 24 Point Design With of 4 WP and 3 SP Factors Using the Same Fraction

| | | | | Fold On $A$ | |
|---|---|---|---|---|---|
| $b$ | $c$ | $ad$ | $abcd$ | $d$ | $bcd$ |
| $p$ | $p$ | $p$ | $p$ | $p$ | $pq$ |
| $q$ | $q$ | $q$ | $q$ | $q$ | $pr$ |
| $r$ | $r$ | $r$ | $r$ | $r$ | $qr$ |
| $pqr$ | $pqr$ | $pqr$ | $pqr$ | $pqr$ | $(1)$ |

I=-PQR

I=PQR

# Case 3: 4 WP Factors (A, B, C, D) and 3 SP Factors (P, Q, R)

- I = ABC = BCD = AD =CPQR = ABPQR = BDPQR = ACDPQR

TABLE 7. 24 Point Design With of 4 WP and 3 SP Factors Using Split-Plot Confounding

| | | | | Fold On $A$ | |
|---|---|---|---|---|---|
| $b$ | $c$ | $ad$ | $abcd$ | $d$ | $bcd$ |
| $pq$ | $p$ | $pq$ | $p$ | $p$ | $pq$ |
| $pr$ | $q$ | $pr$ | $q$ | $q$ | $pr$ |
| $qr$ | $r$ | $qr$ | $r$ | $r$ | $qr$ |
| (1) | $pqr$ | (1) | $pqr$ | $pqr$ | (1) |

I=-PQR

I=PQR

# Constructing 24 run designs based on a block design

- Whole plot factors is less than 4
- 4 whole plots with six subplots in each whole plot

3 high values and 3 low values for each factors

1. the number of subplot factors is odd:mix some of the treatments from the positive fraction and some of the treatments from the negative fraction
2. the number of subplot factorsis even:use only treatments from either the positive fraction or negative fraction in each whole plot

- Result:partially balanced incomplete block design

# Case1:2WP factors (A,B) and 4 SP factors (P,Q,R,S)

- I=PQRS (1) pq pr ps qr qs rs pqrs → $C^3_2+1$

  I=-PQRS p q r s pqr pqs prs qrs → $C^4_3$
- Result :all effects of interest are estimable

TABLE 8. 24 Point Design With of 2 WP and 4 SP Factors Based on a BIB

| $a$ | $b$ | $ab$ | $(1)$ |
|---|---|---|---|
| $p$ | $q$ | $pq$ | $pr$ |
| $q$ | $r$ | $pr$ | $ps$ |
| $r$ | $s$ | $ps$ | $qr$ |
| $pqs$ | $pqr$ | $qr$ | $qs$ |
| $prs$ | $pqs$ | $qs$ | $(1)$ |
| $qrs$ | $prs$ | $rs$ | $pqrs$ |

# Case2:3WP factors (A,B,C) and 3 SP factors (P,Q,R,S)

- I=PQR p q r pqr

  I=-PQR (1) pq pr qr → $C^4_3$
- Result :all effects of interest are estimable

TABLE 9. 24 Point Design With of 3 WP and 3 SP Factors Based on a BIB

| $a$ | $b$ | $c$ | $abc$ |
|---|---|---|---|
| $p$ | $p$ | $q$ | $p$ |
| $pq$ | $pr$ | $r$ | $q$ |
| $pqr$ | $pqr$ | $pqr$ | $r$ |
| $qr$ | $qr$ | $pr$ | $pr$ |
| $r$ | $q$ | $pq$ | $pq$ |
| $(1)$ | $(1)$ | $(1)$ | $qr$ |

# Comparing 24-run and 32-run designs

- predicted variance: $x_i'(X'V^{-1}X)^{-1}x_i$

  X is the design matrix

  **V** is a block diagonal matrix with blocks of $\sigma_{wp}^2 J + \sigma_{sp}^2 I$

  $\mathbf{x_i}$ is the ith row of the combined array
- Relative efficiency ratio:

$$100\left[\frac{(32)\left(\max_{i=1,2,\ldots,32}\left[x_i'(X'V^{-1}X)^{-1}x_i\right]\right)}{(24)\left(\max_{i=1,2,\ldots,24}\left[x_i'(X'V^{-1}X)^{-1}x_i\right]\right)}\right]$$

# Comparing 24-run and 32-run designs

2 Whole-Plot Variables and 4 Subplot Variables → I=ABPQRS

| $\sigma_{wp}^2$ | $\sigma_{sp}^2$ | Relative Efficiencies in Percents: Split-plot Confounding | Same Fraction | BIB |
|---|---|---|---|---|
| 1 | 1 | 79.57 | 88.55 | 93.21 |
| 3 | 1 | 79.60 | 88.75 | 83.76 |
| 5 | 1 | 83.62 | 82.20 | 87.16 |

3 Whole-Plot Variables and 3 Subplot Variables → I=ABCPQR

| $\sigma_{wp}^2$ | $\sigma_{sp}^2$ | Relative Efficiencies in Percents: Split-plot Confounding | Same Fraction | BIB |
|---|---|---|---|---|
| 1 | 1 | 91.55 | 91.63 | 91.04 |
| 3 | 1 | 83.80 | 92.89 | 82.69 |
| 5 | 1 | 95.16 | 93.30 | 92.30 |

4 Whole-Plot Variables and 3 Subplot Variables → I=ABCD =ABPQR =CDPQR

| $\sigma_{wp}^2$ | $\sigma_{sp}^2$ | Relative Efficiencies in Percents: Split-plot Confounding | Same Fraction |
|---|---|---|---|
| 1 | 1 | 90.96 | 90.96 |
| 3 | 1 | 88.73 | 88.73 |
| 5 | 1 | 86.95 | 86.95 |

# Example

- 3 whole plot factor (A,B,C)

  3 subplot factor (P,Q,R)
- treatment structure :AxBxCxPxQxR(2x2x2x2x2)
- Plot structure :4/4

  U(U,V ) /W($W_1$,$W_2$)
- strata :N($n_1$,$n_2$)

  =1+ ($n_1$-1)+ $n_1$( $n_2$-1)

  $S_0$ $S_1$ $S_2$

# Example

V= $S_0 \oplus S_1 \oplus S_2$

1 3 12

U V UV $W_1$,$W_2$,$W_1W_2$

$UW_1$, $UW_2$, $UW_1W_2$

$VW_1$,$VW_2$,$VW_1W_2$

$UVW_1$, $UVW_2$, $UVW_1W_2$

=$V_0 \oplus W_t$

- observational unit (plot) and experimental unit:

  U / W ( $W_1$ ,$W_2$ )

  Exp'tal unit

  Obs'nal unit

# Example

- restrictions on the allocation of treatments to plots :

  A,B,C cannot confound with $W_1$,$W_2$

  A,B,C must be confounded with U,V or UV

- design and design key:

  A=U B=V C=UV P=$W_1$ Q=$W_2$ R=$W_1W_2$

- plan and the restrictions on randomization:

  1.apply treatment a,b,c,abc to larger experimental unit

  2.within each whole plot,randomize treatment p,q,r,pqr

# Example

- how treatment and plot effects are confounded :the whole plot factor must be confounded with some of 3 block effects
- Stratum

| stratum | df | Source |
|---|---|---|
| $S_0$ | 1 | mean |
| $S_1$ | 3 | A,B,C |
| $S_2$ | 12 | P,Q,R,AP,AQ,AR,BP,BQ,BR, CP,CQ,CR |

- ANOVA table

# Analysis

TABLE 12. Effects Table for the Example

| Term | Effect | Std Error | t-value | P-value |
|---|---|---|---|---|
| $A$ | 0.312 | 0.6492 | NA | |
| $B$ | 0.593 | 0.6492 | NA | |
| $C$ | −1.531 | 0.7497 | NA | |
| $P$ | −0.156 | 0.7497 | −0.21 | 0.8431 |
| $Q$ | 0.125 | 0.7497 | 0.17 | 0.8741 |
| $R$ | 0.687 | 0.7497 | 0.92 | 0.4012 |
| $P*Q$ | 0.468 | 0.7497 | 0.63 | 0.5593 |
| $P*R$ | 0.093 | 0.7497 | 0.13 | 0.9054 |
| $Q*R$ | −1.562 | 0.7497 | −2.08 | 0.0916 |
| $A*P$ | −0.156 | 0.6492 | −0.24 | 0.8194 |
| $A*Q$ | −0.078 | 0.6492 | −0.12 | 0.9089 |
| $A*R$ | −0.078 | 0.6492 | −0.12 | 0.9089 |
| $B*P$ | −0.375 | 0.6492 | −0.58 | 0.5886 |
| $B*Q$ | 0.421 | 0.6492 | 0.65 | 0.5445 |
| $B*R$ | 0.546 | 0.6492 | 0.84 | 0.4381 |
| $C*P$ | 0.343 | 0.6492 | 0.53 | 0.6192 |
| $C*Q$ | 0.343 | 0.7497 | 0.46 | 0.6658 |
| $C*R$ | −0.593 | 0.7497 | −0.79 | 0.4643 |

NA—Tests on the whole plot factors are not applicable