

Factorial Experiments When Factor Levels Are Not Necessarily Reset

DEREK F.WEBB / JAMES M.LUCAS / JOHN J.BORKOWSKI

指導教授：鄭少為

學生：蘇靖元 9524519

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Outline

- Example for experiments When Factor Levels Are Not Necessarily Reset
- Comparison of completely randomized experiment and a randomized run order and not reset experiment
- Conclusions

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Introduction

- An experiment is called a completely randomized experiment if it has a randomized run order and all factors in the experiment are reset at the beginning of each run (Ganju and Lucas (1997)).

$$X = \begin{matrix} \text{run} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{matrix} \end{matrix} \begin{pmatrix} x_1 & x_2 & x_3 \\ -1 & -1 & -1 \\ -1 & -1 & 1 \\ -1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix}$$

- An experiment with a randomized run order and not reset experiment (RNR experiment).

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Introduction (cont.)

- In industry, the run order of experiment is often randomized, but not all factors are reset from one run to the next when they occur at the same level.
- For example, in an adhesive curing process the temperature of the curing dispenser is difficult to adjust. The temperature is left unadjusted at the current setting (i.e it is not reset). This is the situation we address in this paper.

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An Industrial Example

- This experiment was successful in improving the performance of a wrapper machine. Three factors, deemed important by engineers for creating a tight seal on the bag containing the product, were examined in this experiment. These factors were the **spacing** of the seal crimper, the **speed** at which the machine was run, and the **temperature** of the seal crimper. Because the mechanics of the machine, speed and spacing were physically hard to reset factors while temperature was easy to reset.

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An Industrial Example (cont.)

The matrix in Table 1 displays the run order in which the experiment was carried out. Spacing and speed were not reset. Temperature was reset at the beginning of each run. The response, seal strength, is also listed. The lines represent restrictions on randomization due to spacing and speed not being reset.

TABLE 1. Wrapper Machine Example

Strength	Spacing	Speed	Temp
8.006	0	1	-1
8.170	0	1	1
8.235	0	0	0
8.450	1	0	1
8.110	1	0	-1
9.158	1	-1	0
5.010	1	1	0
5.890	-1	1	0
10.885	-1	-1	0
8.940	-1	0	-1
9.110	-1	0	1
8.090	0	0	0
8.100	0	-1	-1
10.150	0	-1	1
8.195	0	0	0

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An Industrial Example (cont.)

- The experiment was analyzed using the fixed effects model, $y = X\beta + \varepsilon$, where the typical assumption of independent and identically distributed errors was made. All main effects, two-factor interactions, and squared terms were included in the model.
- The experimenters followed the very common industrial practice of using an ordinary least squares analysis, although they knew that the run order was not completely randomized.

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An Industrial Example (cont.)

Another analysis using the mixed model $y = X\beta + Z\alpha + \varepsilon$ was conducted.

For this example, the Z matrix is formed from matrices. That is, $Z = [Z_{\text{spacing}} | Z_{\text{speed}}]$, where ones in the matrices indicate successive runs where factors have the same level and are not reset.

For example, spacing is reset four times, and there are four columns in Z_{spacing} . The first three rows of the first column are ones because for the first three runs of the experiment spacing was not reset.

and

$$Z_{\text{spacing}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$Z_{\text{speed}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

An Industrial Example (cont.)

The SAS output is shown in Table 3.
We note that the variables B_SPACE and B_SPEED are random effects corresponding to the blocks formed by not resetting SPACING and SPEED. These are the blocks associated with the matrices Zspacing and Zspeed

TABLE 3. SAS Output for Wrapper Machine Example					
Effect	Estimate	Std Error	DF	t	Pr > t
INTERCEPT	3.7411	0.7718	1.08	1.80	0.116
SPACING	-0.8012	0.7481	0.95	-0.67	0.829
SPEED	-2.1417	0.1656	1.11	-12.95	0.000
TEMP	1.4853	0.1397	4.00	10.49	0.001
SPACING*SPEED	0.2350	0.1976	4.00	1.19	0.300
SPACING*TEMP	0.0425	0.1976	4.00	0.22	0.840
SPEED*TEMP	0.7788	0.1976	4.00	3.94	0.017
SPACING*SPACING	-1.1178	1.0509	0.95	-1.05	0.469
SPEED*SPEED	0.0850	0.2078	4.00	0.41	0.700
TEMP*TEMP	-0.4711	0.2078	4.00	-2.28	0.081
Covariance Parameter Estimates (REML)					
B_SPACE	1.080				
B_SPEED	0.000				
Residual	0.155				

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Mixed model

- The mixed model $y=X\beta +Zu+\varepsilon$ is used to account for randomization restrictions in the experiment associated with c factors that are not reset. Where $Z=(Z_{x1} | \dots | Z_{xc})$ is the random effects design matrix for c not reset factors and u is the vector of random effects.
- For every factor x_i that is not reset from one run to the next in an L^k experiment, the standard random effects matrix Z_{x_i} is defined as:

$$Z_{x_i} = j_{L^{i-1}} \otimes I_L \otimes j_{L^{k-i}}$$
where L is the number of levels of each factor, k is the number of factors, and $j_{L^{i-1}}$ and $j_{L^{k-i}}$ are column vectors of ones of sizes $L^{i-1} \times 1$ and $L^{k-i} \times 1$, respectively. The symbol $\otimes$ refers to Kronecker multiplication.

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix}.$$

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Mixed model (cont.)

For example, suppose the design matrix X_D and the standard random effects matrices Z_{x1} and Z_{x2} for the 2^3 factorial experiment are

and

$$X_D = \begin{matrix} \text{run} & a_1 & a_2 & a_3 \\ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{bmatrix} & \begin{bmatrix} -1 & -1 & -1 \\ -1 & -1 & 1 \\ -1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix} & Z_{x1} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \end{matrix}$$
$$Z_{x2} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix}$$

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The Expected Prediction Variance

- Experimenters are often interested in using the results of experiments for response prediction.
- An understanding of the properties of the prediction variance of factorial experiment that contain one or more factors that are not reset from one run to the next is valuable in comparing experiments.

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The Expected Prediction Variance (cont.)

- The prediction variance of RNR experiments depends on how many factors are not reset and on where the lack of resetting occurs in the experiment. Therefore, it is not feasible to present all possible prediction variances for factorial experiments.
- Instead, we give the expected prediction variance for L^k factorial experiments, where the expectation is taken with respect to the discrete distribution of all possible L^k run orders.

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Expected Variance-Covariance (EVC) Theorem

- The expected variance-covariance matrix $V_E = E[\text{Var}(y)]$ of an L^k experiment with randomized run order when $x_1, x_2, \dots, x_c$ are factors that are not reset from one run to the next is

$$V_E = \left[\sigma^2 + (1-p) \sum_{i=1}^c \sigma_i^2 \right] \mathbf{I}_n + p \sum_{i=1}^c \sigma_i^2 \mathbf{Z}_{xi} \mathbf{Z}_{xi}',$$

where $p = 2^c / (L^{k-1}(L-1) + 2)$.


$$E[\text{Var}[\hat{\beta}]] = (\mathbf{X}'\mathbf{X})^{-1} \mathbf{X}' E[\text{Var}[\mathbf{y}]] \mathbf{X} (\mathbf{X}'\mathbf{X})^{-1} \\ = (\mathbf{X}'\mathbf{X})^{-1} \mathbf{X}' V_E \mathbf{X} (\mathbf{X}'\mathbf{X})^{-1} \quad (2)$$

$$E[\text{Var}[\hat{y}]] = \mathbf{X} E[\text{Var}[\hat{\beta}]] \mathbf{X}'.$$

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Expected Variance-Covariance (EVC) Theorem (cont.)

- The maximum prediction variance is a convenient criterion to use for comparison and is directly related to optimal design criterion.
- The maximum expected prediction variance of y_{head} , can be written as $\max_{x \in \mathcal{X}} E[\text{Var}[\hat{y}(x)]] = \mathbf{x}_0' E[\text{Var}[\hat{\beta}]] \mathbf{x}_0$, where $\mathbf{x}_0 = [1, \dots, 1]$
- The EVC theorem allows a comparison to be made between RNR and completely randomized experiments based on the maximum prediction variance of the experimental design.

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Compares completely randomized and RNR 2^k factorial experiments

- Table 4 compares completely randomized and RNR 2^k factorial experiments. Three models are considered: main effects; main effects and all 2-factor interactions; and full model.
- for a completely randomized experiment, the number of resets of the factor(s) that are not reset is the number of runs of the experiment
- the average number of resets for an L^k factorial experiment including one reset for the initial setup as $L^{k+1} - L^{k-1}$. For $L=2$, this reduces to $2^{k-1} + 1$. (Mood (1940))

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Compares completely randomized and RNR 2^k factorial experiments (cont.)

- Each model contains c factors that are not reset, where $c \leq k$. The multipliers of σ^2 and σ^2_j for the maximum prediction variance of a randomized design and the maximum expected prediction variance of a RNR design are given, where σ^2_j is the variance associated with not resetting the j factor, where $1 \leq j \leq c$.

Table 4

TABLE 4. AVERAGE NUMBER OF MODELS AND VARIANCE MULTIPLIERS OF THE MAXIMUM PREDICTION VARIANCE

Factors in the Model	Completely Randomized		RNR		
	Variance Multiplier $\sigma^2 = \sigma_j^2$	Avg. # Models	σ^2	σ_j^2	Models
2	4	4e	4	4.00	4e
3	8	8e	8	4.00	8e
4	16	16e	16	4.00	16e
5	32	32e	32	4.00	32e
6	64	64e	64	4.00	64e
7	128	128e	128	4.00	128e
8	256	256e	256	4.00	256e
9	512	512e	512	4.00	512e
10	1024	1024e	1024	4.00	1024e
11	2048	2048e	2048	4.00	2048e
12	4096	4096e	4096	4.00	4096e
13	8192	8192e	8192	4.00	8192e
14	16384	16384e	16384	4.00	16384e
15	32768	32768e	32768	4.00	32768e
16	65536	65536e	65536	4.00	65536e
17	131072	131072e	131072	4.00	131072e
18	262144	262144e	262144	4.00	262144e
19	524288	524288e	524288	4.00	524288e
20	1048576	1048576e	1048576	4.00	1048576e
21	2097152	2097152e	2097152	4.00	2097152e
22	4194304	4194304e	4194304	4.00	4194304e
23	8388608	8388608e	8388608	4.00	8388608e
24	16777216	16777216e	16777216	4.00	16777216e
25	33554432	33554432e	33554432	4.00	33554432e
26	67108864	67108864e	67108864	4.00	67108864e
27	134217728	134217728e	134217728	4.00	134217728e
28	268435456	268435456e	268435456	4.00	268435456e
29	536870912	536870912e	536870912	4.00	536870912e
30	1073741824	1073741824e	1073741824	4.00	1073741824e
31	2147483648	2147483648e	2147483648	4.00	2147483648e
32	4294967296	4294967296e	4294967296	4.00	4294967296e
33	8589934592	8589934592e	8589934592	4.00	8589934592e
34	17179869184	17179869184e	17179869184	4.00	17179869184e
35	34359738368	34359738368e	34359738368	4.00	34359738368e
36	68719476736	68719476736e	68719476736	4.00	68719476736e
37	137438953472	137438953472e	137438953472	4.00	137438953472e
38	274877906944	274877906944e	274877906944	4.00	274877906944e
39	549755813888	549755813888e	549755813888	4.00	549755813888e
40	1099511627776	1099511627776e	1099511627776	4.00	1099511627776e
41	2199023255552	2199023255552e	2199023255552	4.00	2199023255552e
42	4398046511104	4398046511104e	4398046511104	4.00	4398046511104e
43	8796093022208	8796093022208e	8796093022208	4.00	8796093022208e
44	17592186044416	17592186044416e	17592186044416	4.00	17592186044416e
45	35184372088832	35184372088832e	35184372088832	4.00	35184372088832e
46	70368744177664	70368744177664e	70368744177664	4.00	70368744177664e
47	140737488355328	140737488355328e	140737488355328	4.00	140737488355328e
48	281474976710656	281474976710656e	281474976710656	4.00	281474976710656e
49	562949953421312	562949953421312e	562949953421312	4.00	562949953421312e
50	1125899906842624	1125899906842624e	1125899906842624	4.00	1125899906842624e
51	2251799813685248	2251799813685248e	2251799813685248	4.00	2251799813685248e
52	4503599627370496	4503599627370496e	4503599627370496	4.00	4503599627370496e
53	9007199254740992	9007199254740992e	9007199254740992	4.00	9007199254740992e
54	18014398509481984	18014398509481984e	18014398509481984	4.00	18014398509481984e
55	36028797018963968	36028797018963968e	36028797018963968	4.00	36028797018963968e
56	72057594037927936	72057594037927936e	72057594037927936	4.00	72057594037927936e
57	144115188075855872	144115188075855872e	144115188075855872	4.00	144115188075855872e
58	288230376151711744	288230376151711744e	288230376151711744	4.00	288230376151711744e
59	576460752303423488	576460752303423488e	576460752303423488	4.00	576460752303423488e
60	1152921504606846976	1152921504606846976e	1152921504606846976	4.00	1152921504606846976e
61	2305843009213693952	2305843009213693952e	2305843009213693952	4.00	2305843009213693952e
62	4611686018427387904	4611686018427387904e	4611686018427387904	4.00	4611686018427387904e
63	9223372036854775808	9223372036854775808e	9223372036854775808	4.00	9223372036854775808e
64	18446744073709551616	18446744073709551616e	18446744073709551616	4.00	18446744073709551616e
65	36893488147419103232	36893488147419103232e	36893488147419103232	4.00	36893488147419103232e
66	73786976294838206464	73786976294838206464e	73786976294838206464	4.00	73786976294838206464e
67	147573952589676412928	147573952589676412928e	147573952589676412928	4.00	147573952589676412928e
68	295147905179352825856	295147905179352825856e	295147905179352825856	4.00	295147905179352825856e
69	590295810358705651712	590295810358705651712e	590295810358705651712	4.00	590295810358705651712e
70	1180591620717411303424	1180591620717411303424e	1180591620717411303424	4.00	1180591620717411303424e
71	2361183241434822606848	2361183241434822606848e	2361183241434822606848	4.00	2361183241434822606848e
72	4722366482869645213696	4722366482869645213696e	4722366482869645213696	4.00	4722366482869645213696e
73	9444732965739290427392	9444732965739290427392e	9444732965739290427392	4.00	9444732965739290427392e
74	18889465931478580854784	18889465931478580854784e	18889465931478580854784	4.00	18889465931478580854784e
75	37778931862957161709568	37778931862957161709568e	37778931862957161709568	4.00	37778931862957161709568e
76	75557863725914323419136	75557863725914323419136e	75557863725914323419136	4.00	75557863725914323419136e
77	151115727451828646838272	151115727451828646838272e	151115727451828646838272	4.00	151115727451828646838272e
78	302231454903657293676544	302231454903657293676544e	302231454903657293676544	4.00	302231454903657293676544e
79	604462909807314587353088	604462909807314587353088e	604462909807314587353088	4.00	604462909807314587353088e
80	1208925819614629174706176	1208925819614629174706176e	1208925819614629174706176	4.00	1208925819614629174706176e
81	2417851639229258349412352	2417851639229258349412352e	2417851639229258349412352	4.00	2417851639229258349412352e
82	4835703278458516698824704	4835703278458516698824704e	4835703278458516698824704	4.00	4835703278458516698824704e
83	9671406556917033397649408	9671406556917033397649408e	9671406556917033397649408	4.00	9671406556917033397649408e
84	19342813113834066795298816	19342813113834066795298816e	19342813113834066795298816	4.00	19342813113834066795298816e
85	38685626227668133590597632	38685626227668133590597632e	38685626227668133590597632	4.00	38685626227668133590597632e
86	77371252455336267181195264	77371252455336267181195264e	77371252455336267181195264	4.00	77371252455336267181195264e
87	154742504910672534362390528	154742504910672534362390528e	154742504910672534362390528	4.00	154742504910672534362390528e
88	309485009821345068724781056	309485009821345068724781056e	309485009821345068724781056	4.00	309485009821345068724781056e
89	618970019642690137449562112	618970019642690137449562112e	618970019642690137449562112	4.00	618970019642690137449562112e
90	1237940039285380274899124224	1237940039285380274899124224e	1237940039285380274899124224	4.00	1237940039285380274899124224e
91	2475880078570760549798248448	2475880078570760549798248448e	2475880078570760549798248448	4.00	2475880078570760549798248448e
92	4951760157141521099596496896	4951760157141521099596496896e	4951760157141521099596496896	4.00	4951760157141521099596496896e
93	9903520314283042199192993792	9903520314283042199192993792e	9903520314283042199192993792	4.00	9903520314283042199192993792e
94	19807040628566084398385987584	19807040628566084398385987584e	19807040628566084398385987584	4.00	19807040628566084398385987584e
95	39614081257132168796771975168	39614081257132168796771975168e	39614081257132168796771975168	4.00	39614081257132168796771975168e
96	79228162514264337593543950336	79228162514264337593543950336e	79228162514264337593543950336	4.00	79228162514264337593543950336e
97	158456325028528675187087900672	158456325028528675187087900672e	158456325028528675187087900672	4.00	158456325028528675187087900672e
98	316912650057057350374175801344	316912650057057350374175801344e	316912650057057350374175801344	4.00	316912650057057350374175801344e
99	633825300114114700748351602688	633825300114114700748351602688e	633825300114114700748351602688	4.00	633825300114114700748351602688e
100	1267650600228229401496703205376	1267650600228229401496703205376e	1267650600228229401496703205376	4.00	1267650600228229401496703205376e

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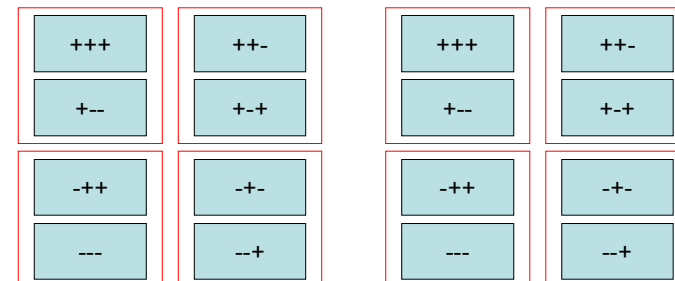
Illustrating the information contained in Table 4

- we consider a 2^4 experiment, where the 11-parameter model contains main effects and all 2-factor interactions with $c = 3$.
- The maximum prediction variance for a completely randomized experiment in which each factor is reset is $(11/16)\sigma^2 + (11/16)\sigma^2_1 + (11/16)\sigma^2_2 + (11/16)\sigma^2_3$
- the maximum expected prediction variance for a RNR experiment is $(11/16)\sigma^2 + (12/16)\sigma^2_1 + (12/16)\sigma^2_2 + (12/16)\sigma^2_3$, which exceeds the maximum prediction variance for the completely randomized experiment by only $(1/16)(\sigma^2_1 + \sigma^2_2 + \sigma^2_3)$
- But the cost of resetting these factors in the completely randomized experiment is $16 \times 3 = 48$, while the RNR experiment requires only $9 \times 3 = 27$.

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Industrial Example Revisited

- We suggest running 2 replicates of a full 2^3 factorial (the 3 factors are spacing (A), speed (B), and temperature (C))
- Each replication of the experiment is blocked on spacing using the relationship $I = A = BC$.



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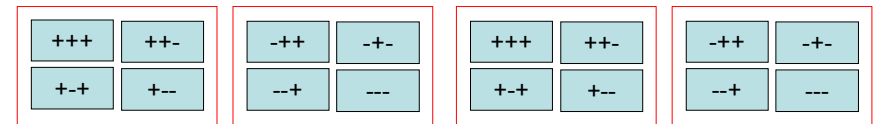
Industrial Example Revisited (cont.)

- It is better than a completely randomized 2 replications of a 2^3 design because it requires that spacing be reset only 8 times.
- Its maximum prediction variance is $(2/8) \sigma^2_A + (7/16) \sigma^2$ while a randomized design has a variance of $(7/16) + (\sigma^2 + \sigma^2_A)$

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Industrial Example Revisited (cont.)

- A second option is to block only on spacing using the relation $I=A$ in each replicate.
- This gives a design with lower cost and less precision because it only requires 4 resets of spacing. Both of these designs are less expensive to run than a completely randomized design, both are easily analyzed, and both have good prediction variance properties.



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Conclusions

- When there are one or two factors that are not reset, a blocked split-plot experiment is better than a randomized experiment because it can have a both lower cost and a smaller prediction variance.

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- Thanks for your attention