Minimum-Aberration Two-Level Fractional Factorial Split-Plot Designs

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Outline

- Two-level Factorial Split-plot Designs
- Minimun Aberration FFSP Designs
- Catalog of Designs
- Search Algorithms
- Testing for Design Isomorphism
- Other Applications of the Combined Approach
- Example with Design Key

Structures of the Designs

- Treatment structure：Fractional Factorial

  Because the restriction of the resource

- Plot Structure：Split Plot

  Because some factors are hard to change

Introduction with a Example

Example 1. Suppose we wish to perform an experiment to identify factors that will improve the efficiency of a ball mill. Engineers have identified seven potential factors, each at two levels:

A. Motor speed
B. Feed mode
C. Feed sizing
D. Material type

→ Hard to change

P. Gain
Q. Screen angle
R. Screen vibration level

$2^{7-3} = 2^{(4+3)-(1+2)}$

D=ABC
Q=P,R=P

Assignment of Factors

- Defining contrast subgroup：

  I=ABCD=PQ=PR=ABCDPQ=ABCDPR=QR=ABCDQR
- Resolution=2
- Better FFSP designs can be constructed using WP factors in the SP fractional generators：

C=AB
Q=DP,R=AP

↓

I=ABC=DPQ=APR=ABCDPQ=BCPR=ADQR=BCDQR

Definition of Minimum Aberration

Definition 1: Minimum Aberration Fractional Factorial Split-Plot. Let $A_i(D)$ denote the number of words of length i in the defining contrast subgroup of an FFSP design, D, and let $A(D) = (A_1(D), A_2(D), \ldots, A_{k_1+k_2}(D))$ be the word-length pattern (WLP) for the FFSP design. Suppose that $D1$ and $D2$ are $2^{(k_1+k_2)-(p_1+p_2)}$ FFSP designs. Let r be the smallest i such that $A_i(D1) \neq A_i(D2)$. Then $D1$ is said to have less aberration than $D2$ if $A_r(D1) < A_r(D2)$. If no such i exists, then $D1$ and $D2$ have equal aberration. A design is said to be MA if no other design has less aberration.

Method of HCV

- Huang, Chen, and Voelkel presented a clever method of decomposing FF designs into FFSP designs.

- Their method falls short of yielding a systematic method for generating all FFSP designs.

Remark

- Not all MA FF designs can be decomposed into FFSP dssigns.

MA 2^{7-3} FF is resolution IV.

But MA $2^{(3+4)-(1+2)}$ FFSP is resolution III

Catalog of Designs

Table 2. Catalog of Minimum Aberration 16-Run FFSP Designs

k	Design	Columns (WP)	Columns (SP)	WLP
5	1.4.0.1		15	0 0 1
5	2.3.0.1		15	0 0 1
5	3.2.0.1		15	0 0 1
6	1.5.0.2		7 11	0 3
6	2.4.0.2		7 11	0 3
6	2.4.0.2		7 13	0 3
6	3.3.0.2		11 12	0 3
6	3.3.1.1	3	13	1 1 1
6	4.2.1.1	7	11	0 3

Table 5. Matrix for 8-run and 16-run Designs

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
0	1	1	0	0	1	1	0	0	1	1	0	0	1	1
0	0	0	1	1	1	1	0	0	0	0	1	1	1	1
0	0	0	0	0	0	0	1	1	1	1	1	1	1	1

NOTE: The independent columns are in boldface and numbered 1, 2, 4, and 8. The 8-run designs use rows 1–3 and the 16-run designs use rows 1–4.

How to use the tables?

- Consider the MA $2^{(3+3)-(1+1)}$ design.

Table 2. Catalog of Minimum Aberration 16-Run FFSP Designs

k	Design	Columns (WP)	Columns (SP)	WLP
6	3.3.1.1	3	13	1 1 1

Table 5. Matrix for 8-run and 16-run Designs

	1	2	3 (C)	4	5	6	7	8	9	10	11	12	13 (R)	14	15
A	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
B	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1
P	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1
Q	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1

NOTE: The independent columns are in boldface and numbered 1, 2, 4, and 8. The 8-run designs use rows 1–3 and the 16-run designs use rows 1–4.

C=AB,R=APQ ⇨ I=ABC=APQR=BCPQR

Remark

- Number of FFSP designs is larger than in the FF case.
- The catalog do not list MA designs of the form $2^{(k1+k2)-(p1+0)}$.
- This catalog is better than HCV's because it shows that how the factors be assigned.

Definition of Isomorphism

Consider two $2^{(k_1+k_2)-(p_1+p_2)}$ FFSP designs, $D1$ and $D2$. Two designs are said to be *isomorphic* if we can obtain $D1$ from $D2$ by relabeling the factors of $D2$. As noted by Chen et al. (1993), isomorphic designs are essentially the same, so it is sufficient to include only one of them in a catalog of designs.

The method of this Paper

- Combined two methods to obtain a more efficient search algorithm:
  1. A search-table approach (Franklin,Baily 1977)
  2. A sequential approach (Chen,J.,Sun,D.X.,Wu,C.F.J 1993)
- Two main features:
  1. The search table eliminates many generators from the search.
  2. At each step we keep only nonisomorpism designs.

A Search-Table Approach

- In a $2^{(k1+k2)-(p1+p2)}$ design,we refer to

  p_1 factors are WP added factors

  p_2 factors are SP added factors

  (k_1-p_1) factors are WP basic factors

  (k_2-p_2) factors are SP basic factors

  The table has $2^{(k1+k2)-(p1+p2)}-(k_1-p_1)-(k_2-p_2)-1$ rows and p_1+p_2 columns.

A example

Let A, B, C, D, and E be the WP factors and P, Q, R, and S be the SP factors. Because of resource constraints, a 16-run $2^{(5+4)-(2+3)}$ FFSP design is desired. Without loss of generality, let D, E, Q, R, and S be the added factors and A, B, C, and P be the basic factors. Consider the constructed search table (Table 4).

Table 4. Search Table

	D	E	Q	R	S
AB	ABD	ABE	—	—	—
AC	ACD	ACE	—	—	—
BC	BCD	BCE	—	—	—
ABC	ABCD	ABCE	—	—	—
AP	—	—	APQ	APR	APS
BP	—	—	BPQ	BPR	BPS
CP	—	—	CPQ	CPR	CPS
ABP	—	—	ABPQ	ABPR	ABPS
ACP	—	—	ACPQ	ACPR	ACPS
BCP	—	—	BCPQ	BCPR	BCPS
ABCP	—	—	ABCPQ	ABCPR	ABCPS

I=ABD=ACE=APQ=BPR=CPS

How many designs?

- There are N_1 x N_2 designs in the table, where

$$N_1=[2^{k_1-p_1}-(k_1-p_1)-1]^{p_1}$$
$$N_2=\{2^{(k_1+k_2)-(p_1+p_2)}-[(k_1+k_2)-(p_1+p_2)]-[2^{k_1-p_1}-(k_1-p_1)-1]-1\}^{p_2}$$

- Franklin noted that selecting from the same row would form designs with resolution less than 3.

AB	ABD	ABE	--		
AC	ACD	ACE	—	—	—
BC	BCD	BCE	—	—	...
ABC	ABCD	ABCE	—	—	—
AP	—	—	APQ	APR	APS
BP	—	—	BPQ	BPR	BPS
CP	-	-	CPQ	CPR	CPS

A Reduction

- We could reduce the number of designs to

$$N_1' \times N_2', where$$

$$N_1' = \prod_{i=1}^{p_1} [2^{k_1-p_1} - (k_1 - p_1) - i]$$

$$N_2' = \prod_{j=1}^{p_2} \{2^{(k_1+k_2)-(p_1+p_2)} - [(k_1+k_2)-(p_1+p_2)] - [2^{k_1-p_1} - (k_1-p_1) - 1] - j\}$$

- This method becomes impractical especially when the run size is large.

A Sequential Approach

Let D_{n_1,n_2,m_1,m_2} be the set of all nonisomorphic $2^{(n_1+n_2)-(m_1+m_2)}$ FFSP designs with resolution $\geq$ III.

Searching for WP designs $\equiv$ Searching for $2^{n_1-m_1}$ FF designs

1. $D_{n_1,0,m_1,0} \Rightarrow D_{n_1+1,0,m_1+1,0}$ by assigning the next factor to an unused column. There are $[2^{n_1-m_1} - (n_1 - m_1) - 1] - m_1$ columns available.
2. $D_{n_1+1,0,m_1+1,0} \Rightarrow D_{n_1+1,0,m_1+1,0}$ by removing designs with resolution III and isomorphism.
3. Keep doing until $D_{k_1,0,p_1,0}$.
4. Use the same procedure to construct D_{k_1,k_2,p_1,p_2}.

The Combined Approach

- The outline of the algorithm is：
  1. Preliminaries
  2. Finding nonisomorphic WP designs
  3. Finding nonisomorphic FFSP designs
- A more detailed version was given by

  Bingham,D.(1998)

  "Design and Analysis of Fractional Factorial Split-Plot Designs"

Finding Nonisomrphic WP Designs

1. Creat $D_{k_1-p_1+1,0,1,0}$ by selecting all length nonisomorphic WP words in the first column.

2. Creat $D_{k_1-p_1+2,0,2,0}$ by selecting a design in $D_{k_1-p_1+1,0,1,0}$ and adding generators from the second column.

3. Remove isomorphic design and go on until $D_{k_1,0,p_1,0}$.

Finding Nonisomorphic FFSP Designs

1. Begin with $D_{k_1,0,p_1,0}$ and use this to construct $D_{k_1,k_2-p_2+1,p_1,1}$.

2. Creat $D_{k_1,k_2-p_2+2,p_1,2}$ by selecting a design in $D_{k_1,k_2-p_2+1,p_1,1}$ and adding generators from the second SP column.

3. Remove isomorphic design and go on until D_{k_1,k_2,p_1,p_2} .

A Example

Suppose we wish to construct $D_{5,4,2,3}$:

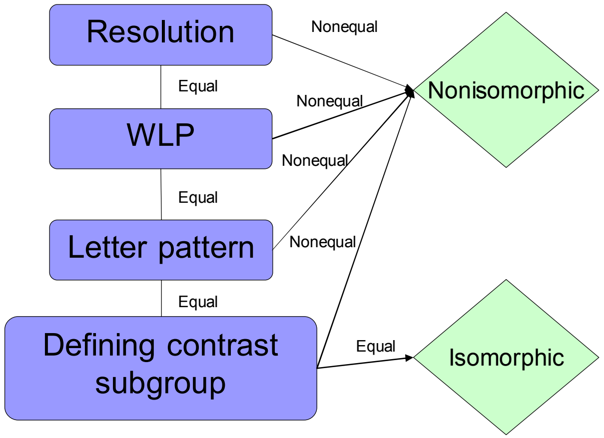
1. Use $D_{4,0,1,0} = \{(ABD),(ABCD)\}$ to construct $D_{5,0,2,0}$.
   (i) If we select generator $g_1 = ABD$.Then the first generator in the second column is $g_2 = ACE$ and add it to $D_{5,0,2,0}$.
   (ii) Move down column 2 and consider $g_2 = BCE$.Then $\{(ABD,BCE)\}$ is isomorphic to $\{(ABD,ACE)\}$. So we discarded it.
   (iii) Do the same procedure and try $g_1 = ABCD$. Finally we get $D_{5,0,2,0} = \{(ABD, ACE),(ABD, ABCE)\}$.

	D	E
AB	ABD	ABE
AC	ACD	ACE
BC	BCD	BCE
ABC	ABCD	ABCE

2. Construct $D_{5,2,2,1}$ from $D_{5,0,2,0}$:
   (i) Choose $g_3 = APQ$ and add it to $D_{5,2,2,1}$.
   (ii) Choose $g_3 = BPQ$ and compare with $\{(ABD,ACE,APQ)\}$.
   (iii) The WLP are different. One is (0,0,3,3,0,0,1) and another is (0,0,3,2,1,1,0).So they are nonisomorphic.Add it into $D_{5,2,2,1}$.
3. Similary , we create $D_{5,3,2,2}$ and $D_{5,4,2,3}$ in this way.

	D	E	Q
AB	ABD	ABE	—
AC	ACD	ACE	—
BC	BCD	BCE	—
ABC	ABCD	ABCE	—
AP	—	—	APQ
BP	—	—	BPQ
CP	—	—	CPQ
ABP	—	—	ABPQ
ACP	—	—	ACPQ
BCP	—	—	BCPQ
ABCP	—	—	ABCPQ

Testing isomorphism



Other Application

- Fractional factorial designs
- q-level fractional factorial designs
- q-level fractional factorial split plot designs

q-Level Fractional Factorial Designs

Example 5. Consider a 3^{4-2} FF design with factors A, B, C, and D. The corresponding search table is shown in Table 6. Notice that the header for each row is a generalized interaction between the factorial effects of the basic factors.

Table 6. Search Table

	C	D
AB	ABC	ABD
A^2B	A^2BC	A^2BD
AB^2	AB^2C	AB^2D
A^2B^2	A^2B^2C	A^2B^2D

$$I = ABC = A^2B^2C^2 = A^2BD = AB^2D^2 = B^2CD = A^2CD = AC^2D^2 = BC^2D^2$$

Example with Design Key

Example 1. Suppose we wish to perform an experiment to identify factors that will improve the efficiency of a ball mill. Engineers have identified seven potential factors, each at two levels:

A. Motor speed
B. Feed mode
C. Feed sizing
D. Material type
P. Gain
Q. Screen angle
R. Screen vibration level

Restrictions and Plan

- Restrictions on the allocation of treatments to plots :

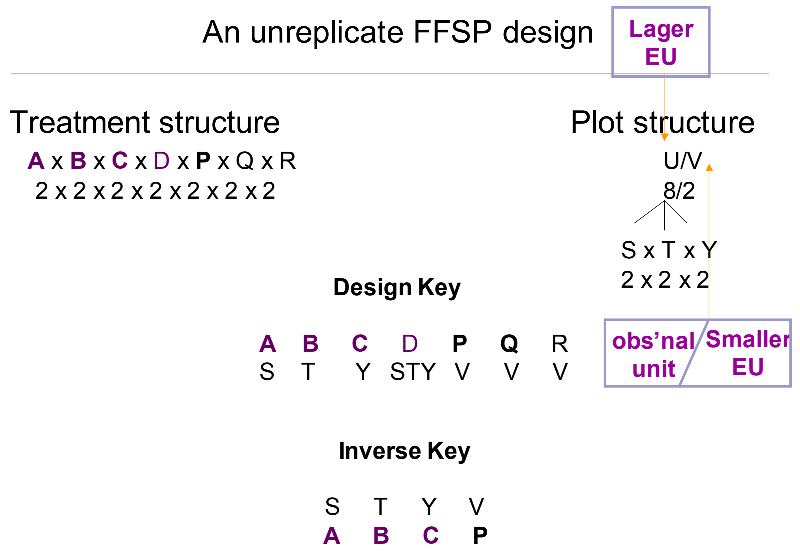
  Treatment main factors can confounded with some of sub-plot effects, but cannot confounded with whole-plot effects.

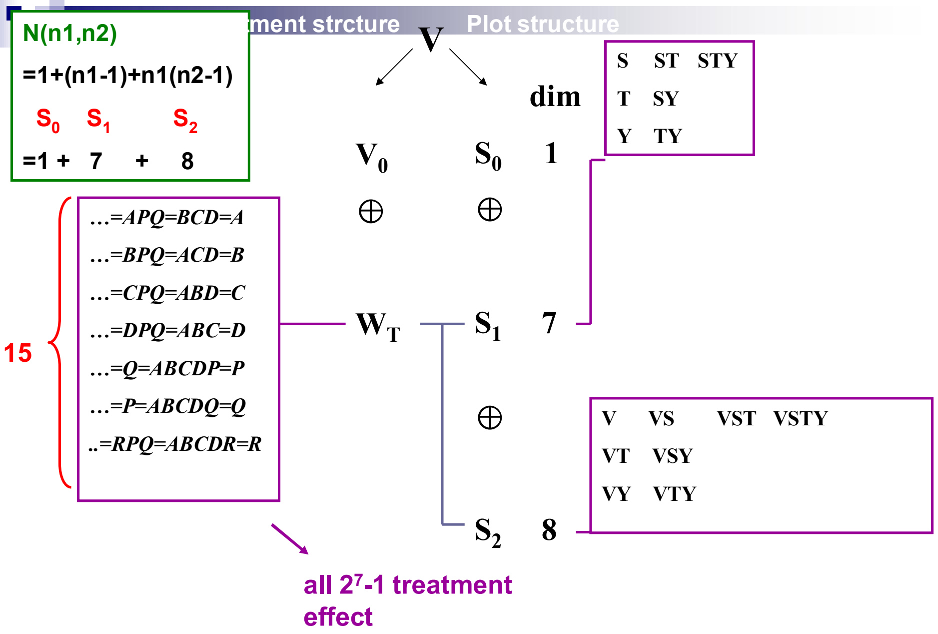
- Plan and the restrictions on randomization:

  Apply treatment A,B,C,D to larger expimential unit.

  Apply trearment P,Q,R to each whole-plot as completely randomized design.

An unreplicate FFSP design





Confounded Sets

I=ABCD=PQ=PR=ABCDPQ=ABCDPR=QR=ABCDQR
A=BCD=APQ=APR=BCDPQ=BCDPR=AQR=BCDQR=S
B=ACD=BPQ=BPR=ACDPQ=ACDPR=BQR=ACDQR=T
C=ABD=CPQ=CPR=ABDPQ=ABDPR=AQR=ABDQR=Y
D=ABC=DPQ=DPR=ABCPQ=ABCPR=DQR=ABCQR=STY
P=ABCDP=Q=R=ABCDQ=ABCDR=PQR=ABCDPQR=V
Q=ABCDQ=P=PQR=ABCDP=ABCDPQR=R=ABCDR=V
R=ABCDR=PQR=P=ABCDPQR=ABCDP=Q=ABCDQ=V

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ANOVA table

stratum	Source	d.f.	S.S.	M.S.
S_0	mean	1	$\frac{\text{sum}^2}{16}$	$\|\tau_0\|^2+\xi_0$
S_1	A,B,C,AB,BC, AC,D	7	$SS(\cdot)$	$\frac{1}{7}\|\tau_w\|^2+\xi_1$
S_2	P,AP,BP,CP, ABP,BCP,ACP, DP	8	$SS(\cdot)$	$\frac{1}{8}\|\tau_s\|^2+\xi_2$
Total		16	$\sum_i y_i^2$	

Thanks For Your Attention