The Design of Blocked Fractional Factorial Split-Plot Experiments

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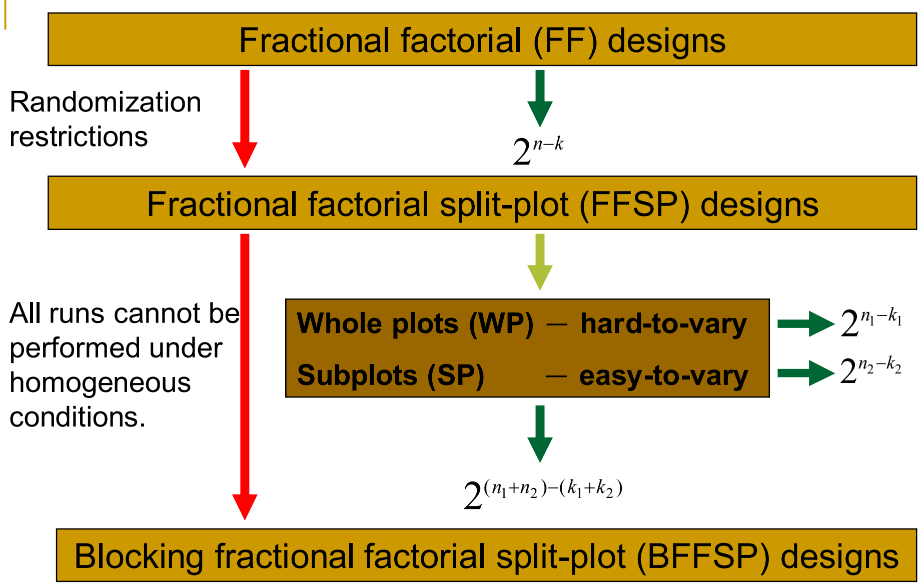
Outline

- Introduction
- Motivating case study
- Three approaches to constructing blocking variables
- Optimality criteria
- A catalog of minimum aberration blocked fractional factorial split-plot designs
- The case study revisited

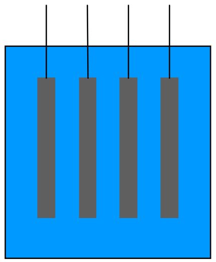


- Blocking may be induced at the WP level using three distinct, yet related, approaches:
  - Pure WP blocking
  - Separation
  - Mixed blocking
- A straightforward extension of the minimum aberration (MA) criterion and other optimality criteria to the BFFSP design setting.
- A catalog of optimal BFFSP designs ranked according to the MA criterion and optimality criteria.

Motivating case study

Chrome-plating experiment from the aerospace sector

A company was experiencing problems with one of its chrome-plating processes in that when a particular complex-shaped part was being plated, excessive pitting and cracking, as well as poor adhesion and uneven deposition of chrome across the part, were observed.



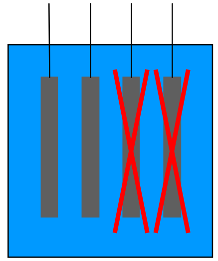
A : chrome concentration
B : chrome to sulfate ratio
C : bath temperature

Hard to vary and changes could be made once per day at most

p : etching current density
q : plating current density
r : part geometry

Easy to vary in that they could be changed at the rectifier level.

There were four rectifiers in the tank, and it was desirable to use only two of these, with the other two used for a separate experiment.



Subplots level → 2^{3-2}

On the positive side, sufficient resources were available to run the experiment for 16 days and to plate two parts per day. These 16 days consisted of four 4-day weeks, and it was desirable to block the experiment by week.

Plot structure : 4/4/2

Three approaches to constructing blocking variables

- The three methods all involve blocking at the WP level, which is the usual goal in blocking a two-level FFSP design.
- Although blocking at the SP level sometimes may be of interest, it is not considered here.

Pure WP blocking

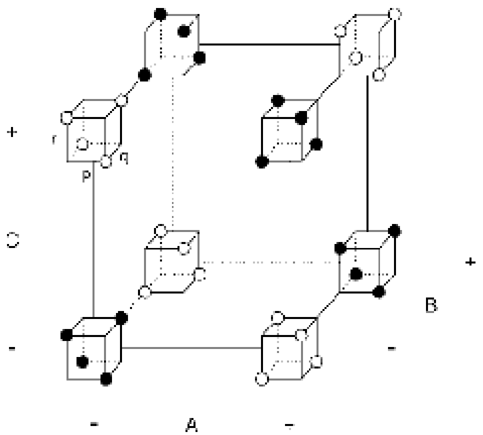
- Pure WP blocking requires that blocking variables be generated exclusively by WP factors.
- Because factor generators and blocking generators are formed simultaneously, the amount of fractionation at the SP level will impact the selection of blocking generators.

For pure WP blocking, the *i*th pure WP blocking variable is denoted by $\beta_i,\ i=1,\ldots,b_1$.
In each block, $2^{n_1-k_1-b_1}$ distinct WP treatment combinations are present, $1 \le b_1 \le n_1 - k_1 - 1$.
Associated with each WP treatment combinations are $2^{n_2-k_2}$ SP treatment combinations.
For compactness of notation, the design is denoted by $2^{(n_1+n_2)-(k_1+k_2)\pm(b_1+0)}$ to refer a two-level BFFSP design having b_1 pure WP-blocking variables.

Example 1

Suppose that we wish to run a $2^{(3+3)-(0+1)}$ FFSP design in $2^1=2$ blocks.
Each block contains four distinct WP treatment combinations and, corresponding to each of the WP treatment combinations, four SP treatment combinations.
Thus there are 16 runs per block.

One possible $2^{(3+3)-(0+1)\pm(1+0)}$ BFFSP design is constructed by using $\beta_1 = ABC$ as the pure WP blocking generator and $r = ABpq$ as the SP factor generator.



2/4/4

Separation

- Separation requires that blocking variables be generated exclusively using SP factors or by using SP factors in tandem with WP factors in blocking generators.
- We call this method of including SP and (possibly) WP factors in the blocking generators “separation,” and call the blocking variables formed in this manner “separators.”

In this form of blocking, the jth separator is denoted by δ_j, $j = 1, \ldots, b_2$.

In each block $2^{n_1 - k_1}$ WP treatment combinations are present, and associated with each WP treatment combination are $2^{n_2 - k_2 - b_2}$ SP treatment combinations, $1 \le b_2 \le n_2 - k_2 - 1$.

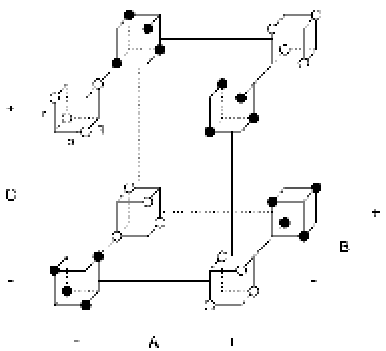
When performing blocking via separation, we refer to the BFFSP design with no WP blocking variables and b_2 separators as a $2^{(n_1+n_2)-(k_1+k_2)\pm(0+b_2)}$ BFFSP design.

Example 2

We return to the $2^{(3+3)-(0+1)}$ FFSP design in Example 1.

Again, we wish to group the 32-run design into two blocks by using one separator instead of one pure WP blocking variable.

One possible $2^{(3+3)-(0+1)\pm(0+1)}$ BFFSP design is formed by using $\delta_1 = ABq$ as the separator and $r = ABCp$ as the SP factor generator.

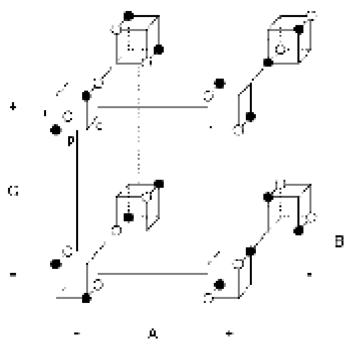


$2^{(3+3)-(0+1)\pm(1+0)}$

$\beta_1 = ABC$

$r = ABpq$

2/4/4



$2^{(3+3)-(0+1)\pm(0+1)}$

$\delta_1 = ABq$

$r = ABCp$

2/8/2

Mixed Blocking

- Mixed blocking is a natural extension of the previous two blocking methods in that we now simultaneously use pure WP blocking variables and separators.

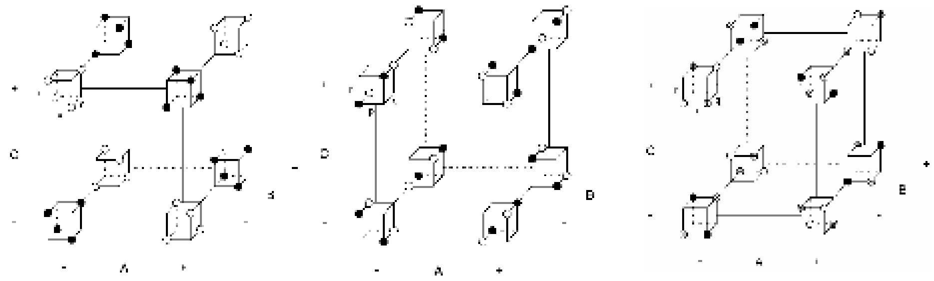
Again, the ith pure WP blocking variable is denoted by β_i and the jth separator is denoted by δ_j.

Under mixed blocking, the formation of b_1 pure WP blocking variables and b_2 separators causes the subsequent $2^{(n_1+n_2)-(k_1+k_2)\pm(b_1+b_2)}$ BFFSP design to be run in $2^{b_1+b_2}$ blocks, where $1 \le b_1 \le n_1 - k_1 - 1$ and $1 \le b_2 \le n_2 - k_2 - 1$.

Example 3

Suppose that we wish to run a $2^{(3+3)-(0+1)}$ design in four blocks by using both a pure WP-blocking variable, β_1, and a separator, δ_1.

One possible $2^{(3+3)-(0+1)\pm(1+1)}$ BFFSP design is formed by using r=ABq as the SP factor generator and β_1=ABC and δ_1=ACpr as the pure WP-blocking generator and separator.



$2^{(3+3)-(0+1)\pm(1+0)}$	$2^{(3+3)-(0+1)\pm(0+1)}$	$2^{(3+3)-(0+1)\pm(1+1)}$
$\beta_1 = ABC$		$\beta_1 = ABC$
	$\delta_1 = ABq$	$\delta_1 = ACqr$
$r = ABpq$	$r = ABCp$	$r = ABq$
2/4/4	**2/8/2**	**4/4/2**

Optimality criteria

1. Minimum aberration blocked fractional factorial split-plot designs
2. A limitation on the word length Definition
3. Additional optimality criteria

- These criteria focus on the ability of these BFFSP designs to estimate lower-order effects.

Sitter et al. (1997) defined the length of a word in the defining contrast subgroup(DCS) of a 2^{n-k} blocked fractional factorial (BFF) design to be

$$\#c_i + (1.5)I_{[\#\beta_i \geq 1]}$$

where $\#c_i$ and $\#\beta_i$ represent the number of factors and blocking variables in the word.

This definition of word length results in the word length pattern of a BFF design of the form

$$W = (A_3, A_{3.5}, \ldots, A_{n+1.5}),$$

where A_i denotes the number of words of length i in the DCS, $3 \leq i \leq n+1.5$.

By extension,we define the length of a word in the DCS of a $2^{(n_1+n_2)-(k_1+k_2)\pm(b_1+b_2)}$ BFFSP design as

$$\#c_i + (1.5)I_{[\#(\beta_i+\delta_j)\geq 1]},$$

where $\#c_i$ and $\#(\beta_i+\delta_j)$ represent the number of factors and blocking variables in the word.

This definition causes theWLP of a BFFSP design to be of the form

$$W = (A_3, A_{3.5}, \ldots, A_{(n_1+n_2)+1.5}),$$

where A_i signifies the number of words of length i in the DCS, $3 \leq i \leq n_1 + n_2 + 1.5$.

Example 4

Consider the $2^{(3+3)-(0+1)\pm(1+1)}$ BFFSP design.

The DCS of the design is $I = ABqr = ABC\beta_1 = ACpr\delta_1 = Cqr\beta_1$ $= BCpq\delta_1 = Bpr\beta_1\delta_1 = Apq\beta_1\delta_1$, which yields the WLP, W=(0, 0, 1, 4, 0, 2).

Definition

Suppose that $D1$ and $D2$ are two $2^{(n_1+n_2)-(k_1+k_2)\pm(b_1+b_2)}$ BFFSP designs.

Let r be the smallest i such that $A_i(D_i) \neq A_i(D_2)$, $3 \leq i \leq n_1 + n_2 + 1.5$.

Then D_1 is said to have less aberration than D_2 if $A_r(D_1) < A_r(D_2)$.

If no such i exists, then D_1 and D_2 are said to have equal aberration.

A BFFSP design is said to be an MA BFFSP design if no other BFFSP design has less aberration.

A limitation on the word length Definition

According to Chen and Cheng (1999), the word length definition of Sitter et al. (1997) has some limitations.

Note that if $A_6 \neq 0$, then a number of three-factor interactions are aliased with other three-factor interactions, whereas if $A_{5.5} \neq 0$, then at least one (less important) four-factor interaction is confounded with blocks.

Because our focus is on the estimation of main effects and two-factor interactions, the definition remains a useful measure for assessing the estimation capability of BFFSP designs.

Additional optimality criteria

- An MA BFFSP design may be further assessed with respect to the following six criteria:
  - (a) The number of clear main effects.
  - (b) The number of clear two-factor interactions.
  - (c) The number of clear SP main effects.
  - (d) The number of clear SP two-factor interactions.
  - (e) The number of clear SP main effects tested against WP error.
  - (f) The number of clear SP two-factor interactions tested against WP error.

A catalog of minimum aberration blocked fractional factorial split-plot designs

All designs have between 7 and 10 factors and blocking variables (combined) and consist of 32 runs in either 2 or 4 blocks.

n_1, n_2	Structure	Design	WLP and generators	(a)	(b)	(c)	(d)	(e)	(f)
4, 4	2:4:4	4, 4; 1, 2; 1, 0	0 2 3 1 4 2 0 2 0 0 0 1 $ABCD$, $AB\beta_1$, ABpqr, ACpqs	8	13	4	13	0	0
	2:8:2	4, 4; 0, 3; 1, 0	0 0 3 3 4 4 0 0 0 0 0 1 $ABD\beta_1$, ABCpq, ACDpr, BDps	8	13	4	8	0	0
	*	*	* $ABCD\beta_1$, ABCpq, ABDpr, ACDps	8	13	4	10	0	3
	2:8:2	4, 4; 1, 2; 0, 1	0 0 3 3 4 4 0 0 0 0 0 1 ABCD, ABpqr, ACpqs, $BCq\delta_1$	8	13	4	13	1	6
	4:2:4	4, 4; 1, 2; 2, 0	0 7 3 2 4 4 0 8 0 1 0 2 ABCD, $AB\beta_1$, $AC\beta_2$, ABpqr, ACpqs	8	13	4	13	0	0
	4:4:2	4, 4; 0, 3; 2, 0	0 1 3 10 4 8 0 0 0 3 0 2 $ABC\beta_1$, $ABD\beta_2$, ACDpq, BCDpr, ABps	8	12	4	8	0	0

4 WPs

4 SPs

4/4/2

$$W = (0,1,3,10,4,8,0,0,0,3,0,2)$$

$$2^{(4+4)-(0+3)\pm(2+0)}$$

BLOCKED FRACTIONAL FACTORIAL SPLIT-PLOT DESIGNS

Table 1. (continued.)

n_1, n_2	Structure	Design	WLP and generators	(a)	(b)	(c)	(d)	(e)	(f)
3, 3	2:4:4	3, 3; 0, 1; 1, 0	0 0 0 1 1 1 $ABC\beta_1$, ABpqr,	6	15	3	12	0	0
	2:8:2	3, 3; 0, 1; 0, 1	0 0 0 1 1 1 ABCqr, $ACpq\delta_1$	6	15	3	12	0	3
	4:2:4	3, 3; 0, 1; 2, 0	0 3 0 0 0 3 1 $AB\beta_1$, $AC\beta_2$, ABCpqr	6	12	3	12	0	0
	4:4:2	3, 3; 0, 1; 1, 1	0 0 1 4 0 2 $ABC\beta_1$, ABqr, $ACpq\delta_1$	6	9	3	7	0	2
	4:4:2	3, 3; 1, 0; 0, 2	1 0 0 3 0 3 ABC, $ABpq\delta_1$, $Bpr\delta_2$	3	12	3	12	0	3
4, 2	2:4:4	4, 2; 1, 0; 1, 0	0 2 1 ABCD, $AB\beta_1$	6	9	2	9	0	0
	2:8:2	4, 2; 0, 1; 1, 0	0 0 0 1 1 1 $ABCD\beta_1$, ABCpq	6	15	2	9	0	1
	4:2:4	4, 2; 1, 0; 2, 0	0 6 1 ABCD, $AB\beta_1$, $AC\beta_2$	6	9	2	9	0	0
	4:4:2	4, 2; 0, 1; 2, 0	0 1 0 3 1 2 $ABC\beta_1$, $ABD\beta_2$, ACDpq	6	14	2	9	0	1

The case study revisited

4 weeks / 4 days / 2 rectifier

Scenario D	Scenario E	Scenario S
r=ABpq	r=ACPq	r=ABq
β_1=ABC	β_1=ABC	β_1=ABC
	β_2=ABP	δ_1=ACpq
2/4/4	**4/4/2**	**4/4/2**

All of the designs use the same number of runs, and all are blocked by week.

The case study revisited

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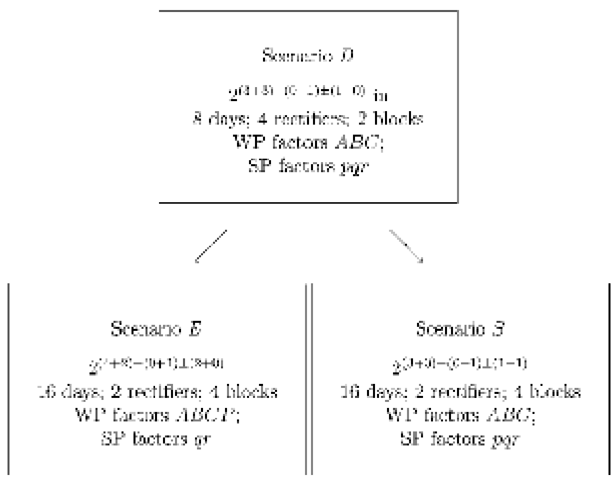

Table 2. Precision of Effect Estimates and Alias Structures for the Three Designs

Effect	Design D	Design E	Design S
A	‡ ‡	‡	†
B	‡ ‡	‡	†
AB	‡ ‡	‡	† qr
C	‡ ‡	‡	†
AC	‡ ‡	‡	†
BC	‡ ‡	‡	†
p		*	
Ap		*	
Bp		*	
Cp		blocks	
q			
Aq			Br
Bq			Ar
Cq			
pq			†
r			
Ar			Bq
Br			Aq
Cr			
pr			†
qr		‡	‡ AB

The presence of at least one asterisk (* or **) implies that the effect of interest is tested against the WP error for that design.

Table 3. Variances of Estimated Effects for Designs D, E, and S

Design	WP effects	SP effects
D	$\frac{1}{2}\sigma_W^2 + \frac{1}{8}\sigma_S^2$	$\frac{1}{8}\sigma_S^2$
E, S	$\frac{1}{4}\sigma_W^2 + \frac{1}{8}\sigma_S^2$	$\frac{1}{8}\sigma_S^2$

Treatment Structure

$A \times B \times C \times P \times q \times r$

$2 \times 2 \times 2 \times 2 \times 2 \times 2$

Experimental units

Plot Structure

$U / V / W$

$u_1 \ u_2 v_1 \ v_2 \ w$

$4/4/2$

Experimental units Observational units

$$N(n_1, N(n_2, n_3))$$
$$= 1 + (n_1 - 1) + n_1(N(n_2, n_3) - 1)$$
$$= 1 + (n_1 - 1) + n_1(1 + (n_2 - 1) + n_2(n_3 - 1) - 1)$$
$$= 1 + (n_1 - 1) + n_1(n_2 - 1) + n_1 n_2 (n_3 - 1)$$

$S_0 \quad S_1 \quad S_2 \quad S_3$

$\mathbf{V}$

$\mathbf{V}_0$ $\mathbf{S}_0$ 1

$\oplus$ $\oplus$

$\mathbf{V}_T$ $\mathbf{S}_1$ u_1, u_2, u_1u_2 3

$\oplus$

$\mathbf{S}_2$ v_1, v_2, v_1v_2 12

$u_1v_1, u_1v_2, u_1v_1v_2$

$u_2v_1, u_2v_2, u_2v_1v_2$

$u_1u_2v_1, u_1u_2v_2, u_1u_2v_1v_2$

$\oplus$

$\mathbf{S}_3$ w 15

u_1w, u_2w, u_1u_2w

v_1w, v_2w, v_1v_2w

$u_1v_1w, u_1v_2w, u_1v_1v_2w$

$u_2v_1w, u_2v_2w, u_2v_1v_2w$

$u_1u_2v_1w, u_1u_2v_2w, u_1u_2v_1v_2w$

Design key

A	B	C	P	q	r
v_1	v_2	$u_1v_1v_2$	$u_2v_1v_2$	w	$u_1u_2v_1w$

Inverse key

u_1	u_2	v_1	v_2	w	I
ABC	ABP	A	B	q	$ACPqr$

$\mathbf{V}$

$\mathbf{V}_0$ $\mathbf{S}_0$ 1

$\oplus$ $\oplus$

$\mathbf{V}_T$ $\mathbf{S}_1$ u_1, u_2, u_1u_2 3 $ABC = BPqr,\ ABP = BCqr,\ CP = Aqr$

$\oplus$

$\mathbf{S}_2$ v_1, v_2, v_1v_2 12 $A = CPqr,\ B = ABCPqr,\ AB = BCPqr$

$u_1v_1, u_1v_2, u_1v_1v_2$ $BC = ABPqr,\ AC = Pqr,\ C = APqr$

$u_2v_1, u_2v_2, u_2v_1v_2$ $BP = ABCqr,\ AP = Cqr,\ P = ACqr$

$u_1u_2v_1, u_1u_2v_2, u_1u_2v_1v_2$ $ACP = qr,\ BCP = ABqr,\ ABCP = Bqr$

$\oplus$

$\mathbf{S}_3$ w 15 $q = AC\text{Pr}$

u_1w, u_2w, u_1u_2w $ABCq = B\text{Pr},\ ABPq = BCr,\ CPq = Ar$

v_1w, v_2w, v_1v_2w $Aq = C\text{Pr},\ Bq = ABC\text{Pr},\ ABq = BC\text{Pr}$

$u_1v_1w, u_1v_2w, u_1v_1v_2w$ $BCq = AB\text{Pr},\ ACq = pr,\ Cq = A\text{Pr}$

$u_2v_1w, u_2v_2w, u_2v_1v_2w$ $BPq = ABCr,\ APq = Cr,\ Pq = ACr$

$u_1u_2v_1w, u_1u_2v_2w, u_1u_2v_1v_2w$ $ACPq = r,\ BCPq = ABr,\ ABCPq = Br$

Stratum	source	df	SS	EMS	VR
mean	mean	1	sum^2 / 32	ξ_0	
large blocks		3	CSS(B)-SS(mean)	ξ_B	
small blocks	A	1	SS(A)	$\lVert\tau_A\rVert^2+\xi_S$	$\lVert\tau_A\rVert^2+\xi_S \big/ \xi_S$
	B	1	SS(B)	$\lVert\tau_B\rVert^2+\xi_S$	$\lVert\tau_B\rVert^2+\xi_S \big/ \xi_S$
	C	1	SS(C)	$\lVert\tau_C\rVert^2+\xi_S$	$\lVert\tau_C\rVert^2+\xi_S \big/ \xi_S$
	P	1	SS(P)	$\lVert\tau_P\rVert^2+\xi_S$	$\lVert\tau_P\rVert^2+\xi_S \big/ \xi_S$
	residual	8		ξ_S	
	total	12	CSS(S)-SS(B)		
plots	q	1	SS(q)	$\lVert\tau_q\rVert^2+\xi$	$\lVert\tau_q\rVert^2+\xi \big/ \xi$
	r	1	SS(r)	$\lVert\tau_r\rVert^2+\xi$	$\lVert\tau_r\rVert^2+\xi \big/ \xi$
	residual	14		ξ	
	total	16	$\sum y^2 - CSS(S)$		
Total		32	$\sum y^2$		

Treatment Structure

$A\times B\times C\times p\times q\times r$

$2\times2\times2\times2\times2\times2$

Experimental units

Plot Structure

$U/V/W$

$u_1 \quad u_2 \, v_1 \quad v_2 \quad w$

$4/4/2$

Experimental units
Observational units

$$N(n_1, N(n_2, n_3))$$
$$= 1 + (n_1 - 1) + n_1(N(n_2, n_3) - 1)$$
$$= 1 + (n_1 - 1) + n_1(1 + (n_2 - 1) + n_2(n_3 - 1) - 1)$$
$$= 1 + (n_1 - 1) + n_1(n_2 - 1) + n_1 n_2 (n_3 - 1)$$

$S_0 \quad S_1 \quad S_2 \quad S_3$

V

$\mathbf{V}_0$ $\mathbf{S}_0$ 1

$\oplus$ $\oplus$

$\mathbf{V}_T$ $\mathbf{S}_1$ $u_1,\ u_2,\ u_1u_2$ 3

$\oplus$

$\mathbf{S}_2$ $v_1,\ v_2,\ v_1v_2$ 12

$u_1v_1,\ u_1v_2,\ u_1v_1v_2$

$u_2v_1,\ u_2v_2,\ u_2v_1v_2$

$u_1u_2v_1,\ u_1u_2v_2,\ u_1u_2v_1v_2$

$\oplus$

$\mathbf{S}_3$ w 15

$u_1w,\ u_2w,\ u_1u_2w$

$v_1w,\ v_2w,\ v_1v_2w$

$u_1v_1w,\ u_1v_2w,\ u_1v_1v_2w$

$u_2v_1w,\ u_2v_2w,\ u_2v_1v_2w$

$u_1u_2v_1w,\ u_1u_2v_2w,\ u_1u_2v_1v_2w$

Design key

A	B	C	p	q	r
v_1	v_2	$u_1v_1v_2$	$u_1u_2v_2w$	w	v_1v_2w

Inverse key

u_1	u_2	v_1	v_2	w	I
ABC	$ACpq$	A	B	q	$ABqr$

V

$\mathbf{V}_0$ $\mathbf{S}_0$ 1

$\oplus$ $\oplus$

$\mathbf{V}_T$ $\mathbf{S}_1$ $u_1,\ u_2,\ u_1u_2$ 3 $ABC = Cqr,\ ACpq = BCpr,\ Bpq = Apr$

$\oplus$

$\mathbf{S}_2$ $v_1,\ v_2,\ v_1v_2$ 12 $A = Bqr,\ B = Aqr,\ AB = qr$

$u_1v_1,\ u_1v_2,\ u_1v_1v_2$ $BC = ACqr,\ AC = BCqr,\ C = ABCqr$

$u_2v_1,\ u_2v_2,\ u_2v_1v_2$ $Cpq = ABCpr,\ ABCpq = Cpr,\ BCpq = ACpr$

$u_1u_2v_1,\ u_1u_2v_2,\ u_1u_2v_1v_2$ $ABpq = pr,\ pq = ABpr,\ Apq = Bpr$

$\oplus$

$\mathbf{S}_3$ w 15 $q = AC\,\mathrm{Pr}$

$u_1w,\ u_2w,\ u_1u_2w$ $ABCq = Cr,\ ACp = BCpqr,\ Bp = Apqr$

$v_1w,\ v_2w,\ v_1v_2w$ $Aq = Br,\ Bq = Ar,\ ABq = r$

$u_1v_1w,\ u_1v_2w,\ u_1v_1v_2w$ $BCq = ACr,\ ACq = BCr,\ Cq = ABCr$

$u_2v_1w,\ u_2v_2w,\ u_2v_1v_2w$ $Cp = ABCpqr,\ ABCp = Cpqr,\ BCp = ACpqr$

$u_1u_2v_1w,\ u_1u_2v_2w,\ u_1u_2v_1v_2w$ $ABp = pqr,\ p = ABpqr,\ Ap = Bpqr$

Stratum	source	df	SS	EMS	VR
mean	mean	1	sum^2 / 32	ξ_0	
large blocks		3	CSS(B)-SS(mean)	ξ_B	
small blocks	A	1	SS(A)	$\|\tau_A\|^2 + \xi_S$	$\|\tau_A\|^2 + \xi_S / \xi_S$
	B	1	SS(B)	$\|\tau_B\|^2 + \xi_S$	$\|\tau_B\|^2 + \xi_S / \xi_S$
	C	1	SS(C)	$\|\tau_P\|^2 + \xi_S$	$\|\tau_C\|^2 + \xi_S / \xi_S$
	residual	9		ξ_S	
	total	12	CSS(S)-SS(B)		
plots	p	1	SS(ρ)	$\|\tau_p\|^2 + \xi$	$\|\tau_p\|^2 + \xi / \xi$
	q	1	SS(q)	$\|\tau_q\|^2 + \xi$	$\|\tau_q\|^2 + \xi / \xi$
	r	1	SS(r)	$\|\tau_r\|^2 + \xi$	$\|\tau_r\|^2 + \xi / \xi$
	residual	13		ξ	
	total	16	$\sum y^2 - CSS(S)$		
Total		32	$\sum y^2$		