Patterns of Confounding in Factorial Designs

R. A. Bailey (1977)
Biometrika, 64, 597-603.

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Outline

- Examples
  - The Design method (design keys)
  - Identification of effects
  - Confounding and aliasing

The Design method

Suppose there are m treatment factors $T_1, \ldots, T_m$. Factor T_i has t_i levels, represented by the integers $0, 1, \ldots, t_i - 1$, for $i = 1, \ldots, m$. Each treatment can be represented by a column vector z with entries $z_1, \ldots, z_m$; z_i gives the level of factor T_i.

Patterson (1976) described the DSIGN method for generating the association of treatments with plots. This uses n plot factors $P_1, \ldots, P_n$, and an $n \times m$ key matrix $K = ((k_{ji}))$ with non-negative integer entries. The plot factors are determined by the block structure; for example, P_1 might represent rows, P_2 columns, P_3 subplots within plots, and so on. Factor P_j has p_j levels, represented by $0, 1, \ldots, p_j - 1$. There is one plot for each combination of levels of the plot factors.

The plot defined by level w_j of factor P_j for $j = 1, \ldots, n$ can be represented by an $n \times 1$ vector w. The DSIGN rule for generating the design is that plot w receives treatment $z(w)$, where $z_i(w) = \Sigma_j k_{ji} w_j \bmod t_i$; that is, $z(w) = K^T w$.

$$z(w)^T = w^T K$$

Treatment structure	Block structure
$A \times B$	$U \times V$
5 5	5 5

Design key

	A	B
	UV	UV^2

Stratum identification

Treatment effect	d.f.	Plot alias	Stratum
A	4	UV	UV
B	4	UV^2	UV
AB	4	U^2V^3	UV
AB^2	4	U^3	U
AB^3	4	U^4V^2	UV
AB^4	4	V^4	V

Inverse key

U	V
A^2B^4	A^4B

Rules of construction

$q(A) = q(U) + q(V) \mod 5$
$q(B) = q(U) + 2q(V) \mod 5$

Design matrix

Block matrix

$$\begin{array}{c} A\ B \\ \begin{bmatrix} 0 & 0 \\ 1 & 2 \\ 2 & 4 \\ 3 & 1 \\ 4 & 3 \\ 1 & 1 \\ 2 & 3 \\ 3 & 0 \\ 4 & 2 \\ 0 & 4 \\ 2 & 2 \\ 3 & 4 \\ 4 & 1 \\ 0 & 3 \\ 1 & 0 \\ 3 & 3 \\ 4 & 0 \\ 0 & 2 \\ 1 & 4 \\ 2 & 1 \\ 4 & 4 \\ 0 & 1 \\ 1 & 3 \\ 2 & 0 \\ 3 & 2 \end{bmatrix} \end{array} = \begin{array}{c} U\ V \\ \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 2 \\ 0 & 3 \\ 0 & 4 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \\ 2 & 0 \\ 2 & 1 \\ 2 & 2 \\ 2 & 3 \\ 2 & 4 \\ 3 & 0 \\ 3 & 1 \\ 3 & 2 \\ 3 & 3 \\ 3 & 4 \\ 4 & 0 \\ 4 & 1 \\ 4 & 2 \\ 4 & 3 \\ 4 & 4 \end{bmatrix} \end{array} \begin{array}{cc} A\ B & \\ \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} & \begin{array}{c} U \\ V \end{array} \end{array}$$

Design key

$$\begin{aligned} z_1(12) &= k_{11}w_1 + k_{12}w_2 \\ &= 1 \cdot 1 + 1 \cdot 2 \\ &\equiv 3 \mod 5 \\ z_2(12) &= k_{21}w_1 + k_{22}w_2 \\ &= 1 \cdot 1 + 2 \cdot 2 \\ &\equiv 0 \mod 5 \end{aligned}$$

Plot structure (5 x 5)

Randomization

Construction of design

Column (level of V)

Row (level of U)	0	1	2	3	4
	Levels of A (first) and B (second)				
0	00	12	24	31	43
1	11	23	30	42	04
2	22	34	41	03	10
3	33	40	02	14	21
4	44	01	13	20	32

An asymmetric design

Treatment structure	*Block structure*
$A \times B \times C$	$X \times Y \times Z$
2 3 6	6 6 2

Design key

A	B	C_1	C_2
$X_1\ Y_1$	$X_2\ Y_2$	$X_1\ Z$	$X_2\ Y_2^2$
2	3	2	3

Stratum contents

Stratum	*d.f.*	*Treatment effects*
X	5	BC_2
Y	5	BC_2^2
Z	1	—
XY	25	A, B, C_2, $A.B$, $A.C_2$, $A.BC_2$, $A.BC_2^2$
XZ	5	C_1, $C_1.BC_2$
YZ	5	AC_1, $AC_1.BC_2^2$
XYZ	25	$B.C_1$, $C_1.C_2$, $C_1.BC_2^2$, $AC_1.B$, $AC_1.C_2$, $AC_1.BC_2$

Rules of construction

$$\begin{aligned} q(A) &= q(X_1)+q(Y_1) && \bmod 2 \\ q(B) &= q(X_2)+q(Y_2) && \bmod 3 \\ q(C_1) &= q(X_1)+q(Z) && \bmod 2 \\ q(C_2) &= q(X_2)+2q(Y_2) && \bmod 3 \end{aligned}$$

Identification of effects

- The bilinear form

$$[z,x] = z^{\mathrm{T}}\Delta_t x \bmod \gamma_t$$

where γ_t is the lowest common multiple of the t_i and Δ_t is the $m \times m$ diagonal matrix with element (i,i) given by γ_t/t_i. Then $T(x)$ is the set of all contrasts between treatments with different values of $[z,x]$, and the x vectors essentially index the sets of treatment contrasts, i.e. main effects and interactions.

$$[z,x] = (z_1,z_2,z_3)\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$= 3x_1z_1 + 2x_2z_2 + x_3z_3$$

Example: degrees of freedom

Table 1. *Values of* $[z, (114)^{\mathrm{T}}]$ *in a* $2 \times 3 \times 6$ *arrangement of treatments*

Levels of T_1	Levels of T_2	Levels of T_3: 0	1	2	3	4	5
0	0	0	2	4	0	2	4
0	1	4	0	2	4	0	2
0	2	2	4	0	2	4	0
1	0	0	2	4	0	2	4
1	1	4	0	2	4	0	2
1	2	2	4	0	2	4	0

T(022)

Example: homogeneous

Table 1. *Values of* $[z, (114)^T]$ *in a* $2 \times 3 \times 6$ *arrangement of treatments*

Levels of T_1	Levels of T_2	Levels of T_3: 0	1	2	3	4	5
0	0	0	4	2	0	4	2
0	1	2	0	4	2	0	4
0	2	4	2	0	4	2	0
1	0	3	1	5	3	1	5
1	1	5	3	1	5	3	1
1	2	1	5	3	1	5	3

T(114): T(100)(1), T(022)(2), T_*(114)(2)

Example: orthogonality

Table 1. *Values of* $[z, (114)^T]$ *in a* $2 \times 3 \times 6$ *arrangement of treatments*

Levels of T_1	Levels of T_2	Levels of T_3: 0	1	2	3	4	5
0	0	0 0	4 1	2 2	0 3	4 4	2 5
0	1	2 2	0 3	4 4	2 5	0 0	4 1
0	2	4 4	2 5	0 0	4 1	2 2	0 3
1	0	3 0	1 1	5 2	3 3	1 4	5 5
1	1	5 2	3 3	1 4	5 5	3 0	1 1
1	2	1 4	5 5	3 0	1 1	5 2	3 3

T(011)

Finite Abelian Group

The set of all vectors x forms a finite Abelian group G, addition of two x vectors consisting of addition of the individual x_i followed by reduction mod t_i. Each vector x generates a cyclic subgroup $\langle x\rangle$ of G consisting of $0, x, \ldots, (s-1)x$, where s is the order of x, that is the smallest positive integer such that all elements of sx are zero. Similarly, the set of all vectors z, which index the treatments, forms a finite Abelian group G^* which is isomorphic to G.

The set $T_*(x)$ is defined to be the set of all contrasts in $T(x)$ which are orthogonal to those in $T(s_1 x), \ldots, T(s_k x)$, where $s_1, \ldots, s_k$ are the proper prime divisors of s. Here $T_*(x)$ represents $\phi(s)$ degrees of freedom, where $\phi(s)$ is the number of integers between 1 and s which are coprime to s. If $T_*(x)$ is constructed for one generator x of each cyclic subgroup of G, we obtain a complete set of orthogonal subsets of the treatment degrees of freedom which are homogeneous in the sense that all contrasts lying in any one subset belong to the same main effect or interaction.

The properties of G ensure that we can find a set of $T_*(x)$ that are orthogonal, homogeneous and exhaustive.

Example: cyclic group

Cyclic subgroup $<x>$ of G

$$\langle(105)\rangle = \{(000),(105),(004),(103),(002),(101)\}, s = 6, \phi(s) = 2$$

$$\langle(104)\rangle = \{(000),(104),(002),(100),(004),(102)\}, s = 6, \phi(s) = 2$$

$$\langle(105)\rangle \cap \langle(104)\rangle = \{(000),(002),(004)\}$$

Cyclic subgroup Euler function

$$\langle(100)\rangle = \{(000),(100)\}, s = 2, \phi(s) = 1$$

$$\langle(002)\rangle = \{(000),(002),(004)\}, s = 3, \phi(s) = 2$$

$$\langle(002)\rangle = \langle(004)\rangle, \quad \langle(022)\rangle = \langle(014)\rangle$$

Table 2. *Identification of orthogonal sets of degrees of freedom in a 2 × 3 × 6 arrangement*

Generator x	Order s of x	Degrees of freedom in $T(x)$	Generators of proper subgroups of $\langle x \rangle$	Degrees of freedom in $T_2(x)$
100	2	$A(1)$	—	$A(1)$
003		$C(1)$	—	$C(1)$
103		$AC(1)$	—	$AC(1)$
010	3	$B(2)$	—	$B(2)$
012		$BC(2)$	—	$BC(2)$
014		$BC(2)$	—	$BC(2)$
002		$C(2)$	—	$C(2)$
110	6	$A(1)$; $B(2)$; $AB(2)$	100, 010	$AB(2)$
111		$AC(1)$; $BC(2)$; $ABC(2)$	103, 014	$ABC(2)$
112		$A(1)$; $BC(2)$; $ABC(2)$	100, 012	$ABC(2)$
113		$AC(1)$; $B(2)$; $ABC(2)$	103, 010	$ABC(2)$
114		$A(1)$; $BC(2)$; $ABC(2)$	100, 014	$ABC(2)$
115		$AC(1)$; $BC(2)$; $ABC(2)$	103, 012	$ABC(2)$
101		$C(2)$; $AC(2)$	103, 002	$AC(2)$
102		$A(1)$; $C(2)$; $AC(2)$	100, 002	$AC(2)$
011		$C(1)$; $BC(4)$	003, 014	$BC(2)$
013		$C(1)$; $B(2)$; $BC(2)$	003, 010	$BC(2)$
015		$C(1)$; $BC(1)$	003, 012	$BC(2)$
001		$C(5)$	003, 002	$C(2)$

$$\langle (114) \rangle = \{(000),(114),(022),(100),(014),(122)\}$$

An asymmetric design

Treatment structure	*Block structure*
$A \times B \times C$	$X \times Y \times Z$
2 3 6	6 6 2

Design key

A	B	C_1	C_2
$X_1\ Y_1$	$X_2\ Y_2$	$X_1\ Z$	$X_2\ Y_2^2$
2	3	2	3

Stratum contents

Stratum	*d.f.*	*Treatment effects*
X	5	BC_2
Y	5	BC_2^2
Z	1	—
XY	25	$A, B, C_2, A.B, A.C_2, A.BC_2, A.BC_2^2$
XZ	5	$C_1, C_1.BC_2$
YZ	5	$AC_1, AC_1.BC_2^2$
XYZ	25	$B.C_1, C_1.C_2, C_1.BC_2^2, AC_1.B, AC_1.C_2, AC_1.BC_2$

Rules of construction

$$q(A) = q(X_1)+q(Y_1) \quad \text{mod } 2$$
$$q(B) = q(X_2)+q(Y_2) \quad \text{mod } 3$$
$$q(C_1) = q(X_1)+q(Z) \quad \text{mod } 2$$
$$q(C_2) = q(X_2)+2q(Y_2) \quad \text{mod } 3$$

Confounding and aliasing

Treatment effect $T(x)$ is aliased with, and estimated by, plot effect $P(y)$ if $[z(w), x] = [w, y]$ for each plot w. We write $T(x) \equiv P(y)$. Bailey, Gilchrist & Patterson (1977) have shown that if the entries of K are such that $k_{ij}p_j/t_i$ is an integer for each i, j, then we can define the integer-entry matrix $K_* = \Delta_p^{-1} K \Delta_t$ and $T(x)$ is estimated by $P(K_* x)$; it is easy to show that $[z(w), x] = [w, K_* x]$ for each plot w. Henceforth we assume that K satisfies this condition: thus we are excluding some of the designs described by Patterson (1976) in order that a simple theory of confounding may be established for the remainder.

$$\alpha(x) = Z^T \Delta_t x \mod \gamma_t$$
$$\beta(y) = W^T \Delta_p y \mod \gamma_p$$

Since $\gamma_p \alpha(x) = \gamma_t \beta(y)$ and $Z^T = W^T K$,

$$y = K_* x,$$

where $K_* = \Delta_p^{-1} K \Delta_t$.

Asymmetric design (nesting)

- Treatment structure: A x B x C (6x6x6)
- Plot structure: X/Y (6/36)
- Some simplification:

Let $\Delta_p = \Delta_t = I$, then $K = K_*$.

$$K = K_* = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 1 & 1 & 3 \end{bmatrix}, \quad K_*^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 4 \\ 5 & 5 & 1 \end{bmatrix}.$$

Confounded effects

$$\langle(125)\rangle = \{(000),(125),(244),(303),(422),(541)\}, s=6, \phi(s)=2$$

$$y = K_* x, \quad \langle(244)\rangle = \{(000),(244),(422)\}, s=3, \phi(s)=2$$

$$x = K_*^{-1} y \quad \langle(125)\rangle = \{(000),(303)\}, s=2, \phi(s)=1$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 4 \\ 5 & 5 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 0 & 2 & 4 \\ 5 & 4 & 3 & 2 & 1 \end{bmatrix}$$

$T_1T_2T_3$ interaction $T_*(125)$ with 2 d.f.

$T_1T_2T_3$ interaction $T_*(244)$ with 2 d.f.

T_1T_3 interaction $T_*(303)$ with 1 d.f.

Asymmetric design (crossing)

- Treatment structure: A x B x C (6x6x6)
- Plot structure: (X x Y) / Z ((6x6)/6)
- Stratum:
  - Rows (plot effects: P(a,0,0))
  - Columns (plot effects: P(0,b,0))
  - Rows x columns (plot effects: P(a,b,0))
  - Subplots (all other plot effects except the mean)

Confounded effects

$$y = (a\ b\ 0)^T \ (a, b = 0, 1, \ldots, 5)$$

$a \backslash b$	0	1	2	3	4	5
0		035	004	033	002	031
1	125	154	123	152	121	150
2	244	213	242	211	240	215
3	303	332	301	330	305	334
4	422	451	420	455	424	453
5	541	510	545	514	543	512

Main effect $T_*(002)$ with 2 d.f.

T_2T_3 interaction $T_*(031)$ with 2 d.f.

T_2T_3 interaction $T_*(033)$ with 1 d.f.

$T_1T_2T_3$ interaction $T_*(154)$ with 18 d.f.

T_1T_2 interaction $T_*(301)$ with 5 d.f.

T_2T_3 interaction $T_*(510)$ with 2 d.f.

Half-replicate

- Treatment structure: A x B x C (2x4x6)
- Plot structure: X/Y (2/12)

$$K = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \end{bmatrix}, \ \Delta_p = \begin{bmatrix} 6 & 0 \\ 0 & 1 \end{bmatrix}, \ \Delta_t = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$K_* = \Delta_p^{-1} K \Delta_t = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 2 \end{bmatrix}.$$

$$x_0 = (123)^T, \quad \alpha(x_0) = Z^T \Delta_t x_0 \mod 12.$$

- T(123) (plot alias P(0)) is not estimable.
- $x^T=(1\ 0\ 0)$, $y^T=(1\ 0)$, $K_* x=y$
- T(100), T(023) confounded with blocks.

Thanks for your attention

With the authors' compliments

References

- Bailey, R. A., Gilchrist, H. L. and Patterson, H. D. (1977) Identification of Effects and Confounding Patterns in Factorial Designs, *Biometrika, 64, 347-354.*

Finite Abelian Group

Definition 1.1.1 一個集合 G 若元素間有一個運算 * 且符合下列性質則稱為一個 group

(GP1)
若 $a, b \in G$ 則 $a*b \in G$

(GP2)
若 $a, b, c \in G$ 則 $(a*b)*c = a*(b*c)$.

(GP3)
在 G 中存在一個元素 e 使得 G 中所有元素 g 都有 $g*e = e*g = g$

(GP4)
對 G 每一元素 g 都可在 G 中找到某一元素 g' 使得 $g*g' = g'*g = e$.

若 G 是一個 group 且對任意的 $a, b \in G$ 都滿足 $a*b = b*a$, 則稱 G 為一個 abelian group.