**Patterns of Confounding in Factorial Designs**

R. A. Bailey (1977)  
*Biometrika, 64, 597-603.*

Reporter: Chien-Yu Peng  
Date: 2007/06/20

---

**Outline**

- Examples
  - The Design method (design keys)
  - Identification of effects
  - Confounding and aliasing

---

**The Design method**

Suppose there are \( m \) treatment factors \( T_1, \ldots, T_m \). Factor \( T_j \) has \( t_j \) levels, represented by the integers \( 0, 1, \ldots, t_j - 1 \), for \( j = 1, \ldots, m \). Each treatment can be represented by a column vector \( x \) with entries \( x_1, \ldots, x_{t_j} \), giving the level of factor \( T_j \).

Paterson (1976) described the method for generating the association of treatments with plots. This uses a plot factors \( P_1, \ldots, P_n \), and an \( n \times m \) key matrix \( K = (k_{ij}) \) with non-negative integer entries. The plot factors are determined by the block structure; for example, \( P_1 \) might represent rows, \( P_2 \) columns, \( P_3 \) subplots within plots, and so on. Factor \( P_j \) has \( p_j \) levels, represented by \( 0, 1, \ldots, p_j - 1 \). There is one plot for each combination of levels of the plot factors.

The plot defined by level \( w_j \) of factor \( P_j \) for \( j = 1, \ldots, n \) can be represented by an \( n \times 1 \) vector \( w \). The design rule for generating the design is that plot \( w \) receives treatment \( x(w) \), where \( x(w) = \sum k_{ij} w_i \mod t_j \) that is, \( x(w) = K w \).

Let \( z(w) = w^T K \).
Plot structure (5 x 5)

**Construction of design**

**Column (level of V)**

<table>
<thead>
<tr>
<th>Row (level of U)</th>
<th>Levels of A (first) and B (second)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>00, 12, 24, 31, 43</td>
</tr>
<tr>
<td>1</td>
<td>11, 23, 30, 42, 04</td>
</tr>
<tr>
<td>2</td>
<td>22, 34, 41, 03, 10</td>
</tr>
<tr>
<td>3</td>
<td>33, 40, 02, 14, 21</td>
</tr>
<tr>
<td>4</td>
<td>44, 01, 13, 20, 32</td>
</tr>
</tbody>
</table>

Randomization

Identification of effects

- The bilinear form

\[
[z, x] = z^T \Delta_t x \mod \gamma_t
\]

where \( \gamma_t \) is the lowest common multiple of the \( t_i \) and \( \Delta_t \) is the \( m \times m \) diagonal matrix with element \((i, i)\) given by \( \gamma_i/t_i \). Thus \( T(z) \) is the set of all contrasts between treatments with different values of \([z, x]\), and the \( r \) vector essentially indexes the sets of treatment contrasts, i.e., main effects and interactions.

\[
[z, x] = (z_1, z_2, z_3)
\]

\[
= 3x_1z_1 + 2x_2z_2 + x_3z_3
\]

Example: degrees of freedom

Table 1. Values of \([z, (1,4)^T]\) in a \(2 \times 3 \times 6\) arrangement of treatments

<table>
<thead>
<tr>
<th>Levels of ( T_1 )</th>
<th>Levels of ( T_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Levels of ( T_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
</tr>
<tr>
<td>1</td>
</tr>
<tr>
<td>2</td>
</tr>
<tr>
<td>3</td>
</tr>
<tr>
<td>4</td>
</tr>
<tr>
<td>5</td>
</tr>
</tbody>
</table>

\[ (0, 1, 1, 2, 2, 2) \]

made by 彭建育
Example: homogeneous

Table 1. Values of [z, (114)\textsuperscript{T}] in a 2 \times 3 \times 6 arrangement of treatments

<table>
<thead>
<tr>
<th>Levels of T\textsubscript{1}</th>
<th>Levels of T\textsubscript{2}</th>
<th>Levels of T\textsubscript{3}</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>5</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>5</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>1</td>
<td>5</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>5</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>5</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>3</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>5</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>5</td>
<td>4</td>
<td>1</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
<td>2</td>
</tr>
</tbody>
</table>

T(114): T(100)(1), T(022)(2), T(114)(2)

Example: orthogonality

Finite Abelian Group

The set of all vectors \( z \) forms a finite Abelian group \( G \), addition of two \( z \) vectors consisting of addition of the individual \( z \), followed by reduction mod \( q \). Each vector \( z \) generates a cyclic subgroup \( \langle z \rangle \) of \( G \) consisting of \( 0, z, \ldots, (q-1)z \), where \( z \) is the order of \( z \), that is the smallest positive integer such that all elements of \( \langle z \rangle \) are zero. Similarly, the set of all vectors \( z \), which index the treatments, forms a finite Abelian group \( G^* \) which is isomorphic to \( G \).

The set \( \mathcal{H}(z) \) is defined to be the set of all contrasts in \( T(z) \) which are orthogonal to those in \( T(z), T(z), T(z), \ldots, T(z) \), where \( z_1, \ldots, z_n \) are the proper prime divisors of \( z \). Then \( \mathcal{H}(z) \) represents \( \phi(q) \) degrees of freedom, where \( \phi(q) \) is the number of integers between 1 and \( q \) which are coprime to \( q \). If \( \mathcal{H}(z) \) is calculated for one generator \( z \) of each cyclic subgroup of \( G \), we obtain a complete set of orthogonal subsets of the treatment degrees of freedom which are homogeneous in the sense that all contrasts lying in any one subset belong to the same main effect of interaction.

The properties of \( G \) ensure that we can find a set of \( \mathcal{H}(z) \) that are orthogonal, homogeneous and exhaustive.

Example: cyclic group

\[ \langle 105 \rangle = \{ (000), (105), (004), (103), (002), (101) \}, s = 6, \phi(s) = 2 \]
\[ \langle 104 \rangle = \{ (000), (104), (002), (100), (004), (102) \}, s = 6, \phi(s) = 2 \]
\[ \langle 105 \rangle \cap \langle 104 \rangle = \{ (000), (002), (004) \} \]

Euler function

\[ \langle 100 \rangle = \{ (000), (100) \}, s = 2, \phi(s) = 1 \]
\[ \langle 002 \rangle = \{ (000), (002), (004) \}, s = 3, \phi(s) = 2 \]
\[ \langle 002 \rangle = \{ (004) \}, \{ (022) \} = \{ (014) \} \]
Table 1. Identification of orthogonal sets of degrees of freedom in a 2 x 3 x 6 arrangement

<table>
<thead>
<tr>
<th>Generator</th>
<th>Order of x</th>
<th>Degrees of freedom in T*(x)</th>
<th>Degrees of freedom in Z*(x)</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>2</td>
<td>A(1)</td>
<td>0</td>
</tr>
<tr>
<td>000</td>
<td>3</td>
<td>E(2)</td>
<td>2</td>
</tr>
<tr>
<td>011</td>
<td>4</td>
<td>E(2)</td>
<td>2</td>
</tr>
<tr>
<td>001</td>
<td>2</td>
<td>A(1)</td>
<td>0</td>
</tr>
<tr>
<td>110</td>
<td>6</td>
<td>A(1); B(2); AB(3)</td>
<td>6</td>
</tr>
<tr>
<td>110</td>
<td>6</td>
<td>C(2)</td>
<td>2</td>
</tr>
<tr>
<td>112</td>
<td>6</td>
<td>A(1); B(2); ABC(3)</td>
<td>6</td>
</tr>
<tr>
<td>112</td>
<td>6</td>
<td>C(2); ABC(3)</td>
<td>4</td>
</tr>
<tr>
<td>113</td>
<td>6</td>
<td>A(1); B(2); ABC(3)</td>
<td>6</td>
</tr>
<tr>
<td>113</td>
<td>6</td>
<td>C(2); ABC(3)</td>
<td>4</td>
</tr>
<tr>
<td>114</td>
<td>6</td>
<td>A(1); B(2); ABC(3)</td>
<td>6</td>
</tr>
<tr>
<td>114</td>
<td>6</td>
<td>C(2); ABC(3)</td>
<td>4</td>
</tr>
<tr>
<td>101</td>
<td>6</td>
<td>A(1); B(2); ABC(3)</td>
<td>6</td>
</tr>
<tr>
<td>102</td>
<td>6</td>
<td>C(2); ABC(3)</td>
<td>4</td>
</tr>
<tr>
<td>011</td>
<td>6</td>
<td>A(1); B(2); ABC(3)</td>
<td>6</td>
</tr>
<tr>
<td>011</td>
<td>6</td>
<td>C(2); ABC(3)</td>
<td>4</td>
</tr>
<tr>
<td>013</td>
<td>6</td>
<td>A(1); B(2); ABC(3)</td>
<td>6</td>
</tr>
<tr>
<td>013</td>
<td>6</td>
<td>C(2); ABC(3)</td>
<td>4</td>
</tr>
</tbody>
</table>

\[
\{114\} = \{(000),(114),(022),(100),(014),(122)\}

Confounding and aliasing

Treatment effect \( T(x) \) is aliased with plot effect \( P(y) \) if \( [a(x), c] = [c, y] \) for each plot \( w \). We write \( T(x) = P(y) \). Bailey, Gilchrist & Patterson (1977) have shown that if the elements of \( K \) are such that \( c_j = j \) for each \( j \), then we can define the identity matrix \( R_g = \Delta_p^T K \Delta_p \) and \( T(x) \) is estimated by \( P(K_{j, x}) \); it is easy to show that \( [f(x), e] = [e, K_{g, x}] \) for each plot \( w \). Henceforth we assume that \( K \) satisfies this condition; then we are excluding some of the designs described by Patterson (1976) in order that a simple theory of confounding may be established for the remainder.

\[
\alpha(x) = Z^T \Delta_p x \mod \gamma_p
\]
\[
\beta(y) = W^T \Delta_p y \mod \gamma_p
\]

Since \( \gamma_p \alpha(x) = \gamma_p \beta(y) \) and \( Z^T = W^T K \),
\[
y = K x,
\]
where \( K = \Delta_p^{-1} K \Delta_p \).

Asymmetric design (nesting)

- Treatment structure: A x B x C (6x6x6)
- Plot structure: X/Y (6/36)
- Some simplification:

Let \( \Delta_p = \Delta_i = I \), then \( K = K_s \).

\[
K_s = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 1 & 3 \end{bmatrix}, \quad K_s^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}
\]

made by 彭建育
Confounded effects

\[
y = K \cdot x,
\]
\[
x = K^{-1} \cdot y
\]

\[
\begin{bmatrix}
1 & 0 & 0 \\
2 & 3 & 4 \\
5 & 5 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 2 & 3 & 4 & 5 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
T_1 T_2 T_3 \text{ interaction } T_3(125) \text{ with 2 d.f.}
\]

\[
T_1 T_2 T_3 \text{ interaction } T_3(244) \text{ with 2 d.f.}
\]

\[
T_1 T_3 \text{ interaction } T_3(303) \text{ with 1 d.f.}
\]

Asymmetric design (crossing)

- Treatment structure: A x B x C (6x6x6)
- Plot structure: (X x Y) / Z ((6x6)/6)
- Stratum:
  - Rows (plot effects: P(a,0,0))
  - Columns (plot effects: P(0,b,0))
  - Rows x columns (plot effects: P(a,b,0))
  - Subplots (all other plot effects except the mean)

Half-replicate

- Treatment structure: A x B x C (2x4x6)
- Plot structure: X/Y (2/12)

\[
K = \begin{bmatrix}
1 & 2 & 3 \\
0 & 1 & 1
\end{bmatrix},
\Delta_p = \begin{bmatrix}
6 & 0 \\
0 & 1
\end{bmatrix},
\Delta_r = \begin{bmatrix}
6 & 0 & 0 \\
0 & 0 & 2
\end{bmatrix}
\]

\[
K_r = \Delta_p^T K \Delta_r = \begin{bmatrix}
1 & 1 & 1 \\
0 & 3 & 0
\end{bmatrix}.
\]

\[
x_0 = (123)^T, \quad \alpha(x_0) = Z^T \Delta_r x_0 \mod 12.
\]
T(123) (plot alias P(0)) is not estimable.

\[ x^T = (1 \ 0 \ 0), \ y^T = (1 \ 0), \ K_x = y \]

T(100), T(023) confounded with blocks.

---

References


---

Finite Abelian Group

Definition 1.1.1  一個群G為阿氏群若有一個等式“保留下列表達等式的一個gpwnp

(GP1)

\[ a, b, c \in G \]

(GP2)

\[ a \cdot b \cdot c = a \cdot (b \cdot c) \]

(GP3)

\[ a \cdot e = a \]

(GP4)

\[ a \cdot a^{-1} = e \]


---

Thanks for your attention

With the authors’ compliments