Complex Aliasing

- Alias matrix
  - True model: $\mathbf{Y} = \mathbf{X}_1\beta_1 + \mathbf{X}_2\beta_2 + \varepsilon$, with $E(\varepsilon) = \mathbf{0}$ and $Var(\varepsilon) = \sigma^2\mathbf{I}$, where $\mathbf{I}$ is the identity matrix.
  - Working model: $\mathbf{Y} = \mathbf{X}_1\beta_1 + \varepsilon$, where $\mathbf{X}_1$ is nonsingular.
  - The ordinary least squares estimator of β_1 under the working model is

$$\hat{\beta}_1 = (\mathbf{X}_1^T\mathbf{X}_1)^{-1}\mathbf{X}_1^T\mathbf{Y}$$

  - Under the true model,

$$\begin{aligned} E(\hat{\beta}_1) &= (\mathbf{X}_1^T\mathbf{X}_1)^{-1}\mathbf{X}_1^T E(\mathbf{Y}) \\ &= (\mathbf{X}_1^T\mathbf{X}_1)^{-1}\mathbf{X}_1^T(\mathbf{X}_1\beta_1 + \mathbf{X}_2\beta_2) \\ &= \beta_1 + (\mathbf{X}_1^T\mathbf{X}_1)^{-1}\mathbf{X}_1^T\mathbf{X}_2\beta_2 \end{aligned}$$

    where $(\mathbf{X}_1^T\mathbf{X}_1)^{-1}\mathbf{X}_1^T\mathbf{X}_2\beta_2$ is the bias of $\hat{\beta}_1$ under the true model.
  - We call $L = (\mathbf{X}_1^T\mathbf{X}_1)^{-1}\mathbf{X}_1^T\mathbf{X}_2$ the **alias matrix** because it provides the aliasing coefficients for the estimate $\hat{\beta}_1$.

Complex Aliasing

- Alias matrix - example
  - Consider the $OA(12, 2^7)$ used in the cast fatigue experiment.
  - Suppose that $\mathbf{X}_1$ consists of the 7 main effects.
  - Suppose that $\mathbf{X}_2$ consists of the $\binom{7}{2} = 21$ two-factor interactions.
  - The true model is unidentiable.
  - Under the working model, the main effects estimates are uncorrelated and unbiased. The main effect estimate (say, of A) is

$$\hat{A} = \bar{y}(A = +) - \bar{y}(A = -)$$

  - Under the true model,

$$\begin{aligned} E(\hat{A}) = A \quad & -\tfrac{1}{3}BC - \tfrac{1}{3}BD - \tfrac{1}{3}BE + \tfrac{1}{3}BF - \tfrac{1}{3}BG \\ & +\tfrac{1}{3}CD - \tfrac{1}{3}CE - \tfrac{1}{3}CF + \tfrac{1}{3}CG + \tfrac{1}{3}DE \\ & +\tfrac{1}{3}DF - \tfrac{1}{3}DG - \tfrac{1}{3}EF - \tfrac{1}{3}EG - \tfrac{1}{3}FG \end{aligned}$$

  - For the 12-run design, each main effect has the same property as that for factor A in that it is partially aliased with the 2-factor interactions not involving the main effect and the aliasing coefficients are $1/3$ or $-1/3$.

Complex Aliasing

- Two effects are said to be **partially aliased** if the absolute value of their aliasing coefficient is strictly between 0 and 1.
- The **complex aliasing** refers to the enormous number of effects in $\mathbf{X}_2$ (e.g., 2-factor interactions) that are partially aliased with the effects in $\mathbf{X}_1$ (e.g., main effects).
- By comparison, for 2^{k-p} or 3^{k-p} regular designs,
  - the factorial effects are either orthogonal (i.e., uncorrelated) or fully aliased (i.e., with aliasing coefficient 1 or -1) with each other, and
  - the number of (full) aliases is much smaller.

❖ **Reading**: textbook, 9.1

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Traditional Analysis Strategy

- For designs with complex aliasing, it is difficult to disentangle the large number of aliased effects and to interpret their significance.
- For this reason, such designs were traditionally used for *screening factor effects* only, i.e., to estimate factor main effects but not their interactions.
- These designs are therefore referred to as **screening designs**.
- The validity of the main effect estimates for screening designs depends on the assumption that the interaction effects are negligible, which is often questionable in many practical situations.
- It suggests the need for strategies that allow interactions to be entertained.

❖ **Reading**: textbook, 9.2

Simplification of Complex Aliasing

- The complex aliasing pattern can be greatly simplified under the assumption of *effect sparsity*.
  - Effect sparsity: number of relatively important effects (e.g., main effects and two-factor interactions) is small.
  - By employing the effect sparsity and effect hierarchy principles, one can assume that only few main effects and even fewer two-factor interactions are relatively important.
- With this assumption, the complex aliasing pattern of a design can be greatly simplified. This simplification and the partial aliasing of effects make it possible to estimate some interactions.

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Simplification of Complex Aliasing

- Consider the hypothetical example.
  - Suppose that in the cast fatigue experiment only A, F, G, AB, and FG are significant.
  - If the seven main effect estimates were obtained under the main-effect-only model, their expectations in LNp.4-26 are simplified as follows:

$$
\begin{aligned}
E(\hat{A}) &= A - \tfrac{1}{3}FG \\
E(\hat{B}) &= -\tfrac{1}{3}FG \\
E(\hat{C}) &= -\tfrac{1}{3}AB + \tfrac{1}{3}FG \\
E(\hat{D}) &= -\tfrac{1}{3}AB + \tfrac{1}{3}FG \\
E(\hat{E}) &= -\tfrac{1}{3}AB - \tfrac{1}{3}FG \\
E(\hat{F}) &= F + \tfrac{1}{3}AB \\
E(\hat{G}) &= G - \tfrac{1}{3}AB
\end{aligned}
\qquad (1)
$$