# Orthogonal Main-Effect Plans through Collapsing

- Method of collapsing factors: useful for constructing plans that are not orthogonal arrays but are smaller in run size than what would be required by the use of orthogonal arrays.

Table 6: Construction of $OME(9, 2^1 3^3)$ from $OA(9, 3^4)$ through Collapsing Factor

| Run | Factor $A$ | $B$ | $C$ | $D$ |
|---|---|---|---|---|
| 1 | 0 | 0 | 0 | 0 |
| 2 | 0 | 1 | 2 | 1 |
| 3 | 0 | 2 | 1 | 2 |
| 4 | 1 | 0 | 1 | 1 |
| 5 | 1 | 1 | 0 | 2 |
| 6 | 1 | 2 | 2 | 0 |
| 7 | 2 | 0 | 2 | 2 |
| 8 | 2 | 1 | 1 | 0 |
| 9 | 2 | 2 | 0 | 1 |

$\longrightarrow$

| Run | Factor $A'$ | $B$ | $C$ | $D$ |
|---|---|---|---|---|
| 1 | 0 | 0 | 0 | 0 |
| 2 | 0 | 1 | 2 | 1 |
| 3 | 0 | 2 | 1 | 2 |
| 4 | 1 | 0 | 1 | 1 |
| 5 | 1 | 1 | 0 | 2 |
| 6 | 1 | 2 | 2 | 0 |
| 7 | 1 | 0 | 2 | 2 |
| 8 | 1 | 1 | 1 | 0 |
| 9 | 1 | 2 | 0 | 1 |





# Orthogonal Main-Effect Plans through Collapsing

- Orthogonal main-effect (*OME*) plan: a plan in which all the main effect estimates are orthogonal (i.e., uncorrelated)
- Proportional frequencies
  - For two factors in the design matrix, denoted by $A$ and $B$, let $r$ and $s$ be the number of levels of $A$ and $B$, respectively.
  - Let $n_{ij}$ be the number of observations at level $i$ of $A$ and level $j$ of $B$ in the two-factor projected design.
  - We say that $n_{ij}$ are in *proportional frequencies* if they satisfy

$$n_{ij} = \frac{n_{i\cdot} n_{\cdot j}}{n_{\cdot\cdot}},$$

  where $n_{i\cdot} = \sum_{j=1}^{s} n_{ij}$, $n_{\cdot j} = \sum_{i=1}^{r} n_{ij}$ and $n_{\cdot\cdot} = \sum_{i=1}^{r} \sum_{j=1}^{s} n_{ij}$.

- In the main-effect-only model, the main effect estimates of factor $A$ and of factor $B$ are uncorrelated (orthogonal) if and only if $n_{ij}$ are in proportional frequencies.

# Orthogonal Main-Effect Plans through Collapsing

Table 7: Factors $A'$ and $B$ in $OME(9, 2^1 3^3)$ appear in proportional frequencies, where $n_{ij}$ appear as cell counts with $n_{i.}$ and $n_{.j}$ as margin totals

| $A'$ | $B$ 0 | 1 | 2 | Row Sum |
|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 3 |
| 1 | 2 | 2 | 2 | 6 |
| Column Sum | 3 | 3 | 3 | |





# Orthogonal Main-Effect Plans through Collapsing

- Construction of *OME* plans from *OA*s through collapsing
  - We can replace a factor with $t$ levels in an *OA* by a new factor with $s$ levels, $s < t$, by using a many-to-one correspondence between the $t$ levels and the $s$ levels.

Table 8: Collapsing a Four-Level Factor to a Three-Level Factor

| | | | | |
|---|---|---|---|---|
| 0 | | 0 | | 0 |
| 1 | | 1 | | 1 |
| 2 | $\longrightarrow$ | 1 | or | 2 |
| 3 | | 2 | | 2 |

  - Because the cell frequencies between any two factors of an *OA* are equal, the collapsed factor and any remaining factor in the orthogonal array have proportional frequencies.
  - The method of collapsing factors can be repeatedly applied to different columns of an *OA* to generate a variety of *OME* plans.

# Orthogonal Main-Effect Plans through Collapsing

- Construction of *OME* plans from *OA*s through collapsing - Examples
  - $OA(9,3^4) \longrightarrow OME(9,2^1 3^3) \longrightarrow OME(9,2^2 3^2) \longrightarrow OME(9,2^3 3^1) \longrightarrow OME(9,2^4)$
  - $OA(8,4^1 2^4) \longrightarrow OME(8,3^1 2^4)$
  - $OA(16,4^2 2^9) \longrightarrow OME(16,4^1 3^1 2^9) \longrightarrow OME(16,3^2 2^9)$
- Advantage of *OME* plans over *OA*s: run size economy
- Disadvantage of *OME* plans comparied to *OA*s
  - Imbalance of run size allocation between levels
  - Loss in estimation efficiency
    * But, the estimation efficiency of *OME* plans is generally very high unless the number of levels of the collapsed factor is much smaller than that of the original factor.





# Orthogonal Main-Effect Plans through Collapsing

- Estimation efficiency comparison between *OME* plans and *OA*s
  - Collapsing a 3-level factor to a 2-level factor: assume that the total run size is $3m$ and the run size allocation between the 2 levels is $2m$ and $m$
    * Variance of its main effect estimate is

$$Var(\bar{y}_1 - \bar{y}_0) = \sigma^2(1/m + 1/2m) = (3/2m)\sigma^2.$$

    * For balanced allocation of $\frac{3m}{2}$ runs to each levels, the variance is $\frac{4}{3m}\sigma^2$
    * Relative efficiency: $\frac{8}{9}$
  - Collapsing a 4-level factor to a 3-level factor: relative efficiency for linear main effect is $\frac{3}{4}$, and relative efficiency for quadratic main effect is $\frac{9}{8}$ if the middle level receives one-half of the runs
- *OME* plans constructed through collapsing fewer factors are preferred over those through collapsing more factors because the collapsing of each factor would result in the loss of degrees of freedom and of estimation efficiency. Example: $OME(8,3^1 2^4)$ and the $OME(9,3^1 2^3)$

❖ **Reading**: textbook, 8.8