Nonregular Designs

- Regular designs: designs whose design matrices are constructed through defining relations among factors
  - For regular designs, any two factorial effects either can be estimated independent of each orther or are fully aliased
- Nonregular designs: designs that are not regular designs
- In Tables 1 and 2, the design used does not belong to the 2^{k-p} series (Chapter 5) or the 3^{k-p} series (Chapter 6), because the latter would require run size as a power of 2 or 3. These designs belong to the class of orthogonal arrays.

Example : $OA(12, 2^{11})$

Table 1: Design Matrix and Lifetime Data, Cast Fatigue Experiment

	Factor											Logged
Run	A	B	C	D	E	F	G	8	9	10	11	Lifetime
1	+	+	–	+	+	+	–	–	–	+	–	6.058
2	+	–	+	+	+	–	–	–	+	–	+	4.733
3	–	+	+	+	–	–	–	+	–	+	+	4.625
4	+	+	+	–	–	–	+	–	+	+	–	5.899
5	+	+	–	–	–	+	–	+	+	–	+	7.000
6	+	–	–	–	+	–	+	+	–	+	+	5.752
7	–	–	–	+	–	+	+	–	+	+	+	5.682
8	–	–	+	–	+	+	–	+	+	+	–	6.607
9	–	+	–	+	+	–	+	+	+	–	–	5.818
10	+	–	+	+	–	+	+	+	–	–	–	5.917
11	–	+	+	–	+	+	+	–	–	–	+	5.863
12	–	–	–	–	–	–	–	–	–	–	–	4.809

Example : $OA(18, 2^1 3^7)$

Table 2: Design Matrix and Response Data, Blood Glucose Experiment

Run	Factor								Mean
	A	*G*	*B*	*C*	*D*	*E*	*F*	*H*	Reading
1	0	0	0	0	0	0	0	0	97.94
2	0	0	1	1	1	1	1	1	83.40
3	0	0	2	2	2	2	2	2	95.88
4	0	1	0	0	1	1	2	2	88.86
5	0	1	1	1	2	2	0	0	106.58
6	0	1	2	2	0	0	1	1	89.57
7	0	2	0	1	0	2	1	2	91.98
8	0	2	1	2	1	0	2	0	98.41
9	0	2	2	0	2	1	0	1	87.56
10	1	0	0	2	2	1	1	0	88.11
11	1	0	1	0	0	2	2	1	83.81
12	1	0	2	1	1	0	0	2	98.27
13	1	1	0	1	2	0	2	1	115.52
14	1	1	1	2	0	1	0	2	94.89
15	1	1	2	0	1	2	1	0	94.70
16	1	2	0	2	1	2	0	1	121.62
17	1	2	1	0	2	0	1	2	93.86
18	1	2	2	1	0	1	2	0	96.10

Orthogonal Arrays

- An **orthogonal array** $OA(N, s_1^{m_1} \ldots s_\gamma^{m_\gamma}, t)$ of strength t is an $N \times m$ matrix, $m = m_1 + \ldots + m_\gamma$, in which m_i columns have s_i (≥ 2) symbols or levels such that, for any t columns, all possible combinations of symbols appear equally often in the matrix.
  - A regular design is an OA of strength $R-1$, where R is the resolution of the regular design
  - $\gamma = 1 \Rightarrow$ **symmetrical** OAs; $\gamma > 1 \Rightarrow$ **asymmetric** (or **mixed-level**) OAs
- For OA of strength two, the index $t = 2$ is dropped for simplicity.
- An $OA(12, 2^{11})$ is used in Table 1 and an $OA(18, 2^1 3^7)$ is used in Table 2.

v **Reading**: textbook, 8.1, 8.3

Why Using Orthogonal Array

- **Run size economy.** Suppose 8-11 factors at two levels are to be studied. Using an $OA(12,2^{11})$ will save 4 runs over a 16-run 2^{k-p} design. Similarly, suppose 5-7 factors at three levels are to be studied. Using an $OA(18,3^7)$ will save 9 runs over a 27-run 3^{k-p} design.
- **Flexibility.** Many OAs exist for flexible combinations of factor levels (see the collection of some useful OAs on later slides).
- Analysis strategy for experiments based on OA can be found in Chapter 9 of WH.

v **Reading**: textbook, 8.2

A Lemma on Orthogonal Arrays

- **Lemma.** For an orthogonal array $OA(N, {s_1}^{m_1} \cdots {s_\gamma}^{m_\gamma}, t)$, its run size N must be divisible by the least common multiple of $\prod_{i=1}^{\gamma} {s_i}^{k_i}$, for all possible combinations of k_i with $k_i \le m_i$ and $\sum_{i=1}^{\gamma} k_i = t$, $i = 1, \ldots, \gamma$.
  Proof:
  - According to the definition of OAs, in any t columns of the array with k_i columns having s_i symbols, $k_i \le m_i$, $\sum_{i=1}^{\gamma} k_i = t$, each of the $\prod_{i=1}^{\gamma} {s_i}^{k_i}$ combinations of symbols appears equally often.
  - N is divisible by the least common multiple of $\prod_{i=1}^{\gamma} {s_i}^{k_i}$ over these choices of t columns.
- Examples:

v **Reading**: textbook, 8.3

Isomorphic Designs

- Two *design matrices* are said to be **isomorphic** (or **equivalent**) if one design matrix can be obtained from the other by
  - row permutations,
  - column permutations,
  - relabeling of levels

Hadamard Matrix

- A **Hadamard matrix** of order N, denoted by H_N, is an $N \times N$ orthogonal matrix with entries 1 or -1
  - without loss of generality, we can assume its first column consists of 1's
  - the remaining $N-1$ columns are orthogonal to the first column and must have half 1's and half -1's.
- By removing the first column from a Hadamard matrix, the remaining $N \times (N-1)$ matrix is an $OA(N, 2^{N-1})$.
  - the construction of $OA(N, 2^{N-1})$ is equivalent to the construction of H_N.
  - the $OA(N, 2^{N-1})$ can be used as a design matrix of $N-1$ 2-level factors while the H_N can be used as a model matrix (for the main effect model), whose first column corresponds to the intercept term

Hadamard Matrix

- An integer N is a *Hadamard number* if an H_N exists.
- For any Hadamard number N, H_{2N} can be constructed as follows,

$$H_{2N} = \begin{pmatrix} H_N & H_N \\ H_N & -H_N \end{pmatrix}.$$

- It is straightforward to verify that the H_{2N} matrix is indeed a Hadamard matrix.
- By removing its first column, we then obtain an $OA(2N, 2^{2N-1})$ for any Hadamard number N.
- Up to isomorphism, there is a unique Hadamard matrix for $N = 1, 2, 4, 8$, and 12. There are 5 nonisomrphic matrices of N =16, 3 of N =20, 60 of N =24, and 487 of N =28.

Plackett-Burman Design (PBD)

- A large collection of $OA(N, 2^{N-1})$ was given in Plackett and Burman (1946) for $N = 4k$ but not a power of two.
- These designs can be generated from the first row by *cyclic* arrangement.
  - the next row is generated from the first row by moving the entries of the previous row one position to the right and placing the last entry in the first position
  - the process continues until $N-1$ rows are generated
  - a row of -1's is then added as the last row, completing the design.

$$+ + - + + + - - - + -$$

Plackett-Burman Design (PBD)

Table 3: Generating Row Vectors for Plackett-Bruman Designs of Run Size N

N	Vector
12	$++-+++---+-$
20	$++--++++-+-+----++-$
24	$+++++-+-++--++--+-+----$
36	$-+-+++---+++++-+++--+----+-+-++--+-$
44	$++--+-+--+++-+++++---+-+++-----+---$
	$++-+-++-$

Projection Properties of PBDs

- Why needs to study projection property?
  - Take the 12-run PBD as an example.
  - Suppose the analysis of main-effect-only model suggests that there are only three significant main effects.
  - After projected onto the three factors, can the desgin matrix allow interactions among the three factors to be estimated?
  - The estimability of interactions after projection is referred to as **hidden projection** property
- For the 12-run PBD
  - Any set of three columns of the design consists of the sum of a 2^3 design and a 2^{3-1} design with $\mathbf{I} = -ABC$
  - Any projection onto four columns of the design is isomorphic to the one given in Figure 1.
  - Any projection onto five factors allows all the main effects and five or six two-factor interactions among the five factors to be estimated.

Projection Properties of PBDs

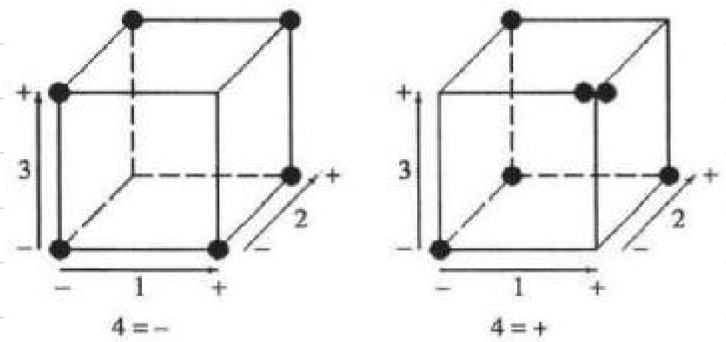

Figure 1: Projection of $OA(12, 2^{11})$ in Appendix 8A.1 (textbook) onto its first four columns

- **Theorem.** (Cheng, 1995) Let **X** be any $OA(N, 2^k)$ with $k \geq 4$ and N not a multiple of 8. Then the projection of **X** onto any four factors has the property that all the main effects and two-factor interactions among the four factors are estimable when the higher-order interactions are negligible.
- More details and results on the hidden projection properties can be found in Lin and Draper (1992) and Wang and Wu (1995)

Hall's Designs

- Hall (1961) proved that there are exactly five nonisomorphic Hadamard matrices for $N = 16$ ($\Rightarrow$ five nonisomorphic $OA(16, 2^{15})$)
- The saturated 2^{15-11}_{III} design is a regular $OA(16, 2^{15})$ and is called a type I design
- The other four nonregular $OA(16, 2^{15})$s are given in Appendix 8B (textbook), in the ascending order of their "nonregularity"
  - The type II design is the closest to being regular in that it has the smallest number of pairs of main effect and two-factor interactions that are partially aliased
  - On the other hand, the type V design is the most nonregular
- The four nonregular $OA(16, 2^{15})$s may be considered as alternatives to the regular one, expecially if the complex aliasing of the former deisngs is considered to be an advantage

v **Reading**: textbook, 8.4

Useful Orthogonal Arrays

- Collection in Appendix 8A and 8C of textbook:

$$*OA(12,2^{11}) \quad OA(12,3^1 2^4) \quad *OA(18,2^1 3^7)$$
$$*OA(18,6^1 3^6) \quad OA(20,2^{19}) \quad OA(24,2^{23})$$
$$OA(24,3^1 2^{16}) \quad OA(24,6^1 2^{14}) \quad *OA(36,2^{11} 3^{12})$$
$$OA(36,3^7 6^3) \quad OA(36,2^8 6^3) \quad OA(48,2^{11} 4^{12})$$
$$OA(50,2^1 5^{11}) \quad OA(54,2^1 3^{25})$$

  * especially useful

- Learn to choose and use the design tables in the collection.
- A library of orthogonal arrays: http://neilsloane.com/oadir/

$OA(18,2^1 3^7)$ and $OA(18,6^1 3^6)$

Table 4: $OA(18,2^1 3^7)$ (columns 1–8) and $OA(18,6^1 3^6)$ (columns 1′ and 3–8)

Run	1′	1	2	3	4	5	6	7	8
1	0	0	0	0	0	0	0	0	0
2	0	0	0	1	1	1	1	1	1
3	0	0	0	2	2	2	2	2	2
4	1	0	1	0	0	1	1	2	2
5	1	0	1	1	1	2	2	0	0
6	1	0	1	2	2	0	0	1	1
7	2	0	2	0	1	0	2	1	2
8	2	0	2	1	2	1	0	2	0
9	2	0	2	2	0	2	1	0	1
10	3	1	0	0	2	2	1	1	0
11	3	1	0	1	0	0	2	2	1
12	3	1	0	2	1	1	0	0	2
13	4	1	1	0	1	2	0	2	1
14	4	1	1	1	2	0	1	0	2
15	4	1	1	2	0	1	2	1	0
16	5	1	2	0	2	1	2	0	1
17	5	1	2	1	0	2	0	1	2
18	5	1	2	2	1	0	1	2	0

$OA(36, 2^{11}3^{12})$

Table 5: $OA(36, 2^{11}3^{12})$

Run	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1
3	0	0	0	0	0	0	0	0	0	0	0	2	2	2	2	2	2	2	2	2	2	2	2
4	0	0	0	0	0	1	1	1	1	1	1	0	0	0	0	1	1	1	1	2	2	2	2
5	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	2	2	2	2	0	0	0	0
6	0	0	0	0	0	1	1	1	1	1	1	2	2	2	2	0	0	0	0	1	1	1	1
7	0	0	1	1	1	0	0	0	1	1	1	0	0	1	2	0	1	2	2	0	1	1	2
8	0	0	1	1	1	0	0	0	1	1	1	1	1	2	0	1	2	0	0	1	2	2	0
9	0	0	1	1	1	0	0	0	1	1	1	2	2	0	1	2	0	1	1	2	0	0	1
10	0	1	0	1	1	0	1	1	0	0	1	0	0	2	1	0	2	1	2	1	0	2	1
11	0	1	0	1	1	0	1	1	0	0	1	1	1	0	2	1	0	2	0	2	1	0	2
12	0	1	0	1	1	0	1	1	0	0	1	2	2	1	0	2	1	0	1	0	2	1	0
13	0	1	1	0	1	1	0	1	0	1	0	0	1	2	0	2	1	0	2	2	1	0	1
14	0	1	1	0	1	1	0	1	0	1	0	1	2	0	1	0	2	1	0	0	2	1	2
15	0	1	1	0	1	1	0	1	0	1	0	2	0	1	2	1	0	2	1	1	0	2	0
16	0	1	1	1	0	1	1	0	1	0	0	0	1	2	1	0	0	2	1	2	2	1	0
17	0	1	1	1	0	1	1	0	1	0	0	1	2	0	2	1	1	0	2	0	0	2	1
18	0	1	1	1	0	1	1	0	1	0	0	2	0	1	0	2	2	1	0	1	1	0	2

$OA(36, 2^{11}3^{12})$

Table 5: Continued

Run	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
19	1	0	1	1	0	0	1	1	0	1	0	0	1	0	2	2	2	0	1	1	0	1	2
20	1	0	1	1	0	0	1	1	0	1	0	1	2	1	0	0	0	1	2	2	1	2	0
21	1	0	1	1	0	0	1	1	0	1	0	2	0	2	1	1	1	2	0	0	2	0	1
22	1	0	1	0	1	1	1	0	0	0	1	0	1	1	2	2	0	1	0	0	2	2	1
23	1	0	1	0	1	1	1	0	0	0	1	1	2	2	0	0	1	2	1	1	0	0	2
24	1	0	1	0	1	1	1	0	0	0	1	2	0	0	1	1	2	0	2	2	1	1	0
25	1	0	0	1	1	1	0	1	1	0	0	0	2	1	0	1	2	2	0	2	0	1	1
26	1	0	0	1	1	1	0	1	1	0	0	1	0	2	1	2	0	0	1	0	1	2	2
27	1	0	0	1	1	1	0	1	1	0	0	2	1	0	2	0	1	1	2	1	2	0	0
28	1	1	1	0	0	0	0	1	1	0	1	0	2	1	1	1	0	0	2	1	2	0	2
29	1	1	1	0	0	0	0	1	1	0	1	1	0	2	2	2	1	1	0	2	0	1	0
30	1	1	1	0	0	0	0	1	1	0	1	2	1	0	0	0	2	2	1	0	1	2	1
31	1	1	0	1	0	1	0	0	0	1	1	0	2	2	2	1	2	1	1	0	1	0	0
32	1	1	0	1	0	1	0	0	0	1	1	1	0	0	0	2	0	2	2	1	2	1	1
33	1	1	0	1	0	1	0	0	0	1	1	2	1	1	1	0	1	0	0	2	0	2	2
34	1	1	0	0	1	0	1	0	1	1	0	0	2	0	1	2	1	2	0	1	1	2	0
35	1	1	0	0	1	0	1	0	1	1	0	1	0	1	2	0	2	0	1	2	2	0	1
36	1	1	0	0	1	0	1	0	1	1	0	2	1	2	0	1	0	1	2	0	0	1	2

- **Reading**: textbook, 8.5