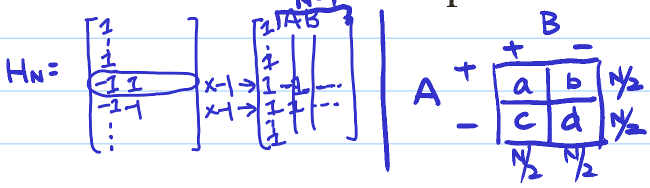
construction of 2-level saturated OA →

# Hadamard Matrix

$H_N^T H_N = H_N H_N^T = N I$

$(H_N)^{-1} = \frac{1}{N} H_N^T$

- A **Hadamard matrix** of order $N$ (run size), denoted by $H_N$, is an $N \times N$ orthogonal matrix with entries 1 or $-1$
  - without loss of generality, we can assume its first column consists of 1's
  - the remaining $N-1$ columns are orthogonal to the first column and must have half 1's and half $-1$'s.
- By removing the first column from a Hadamard matrix, the remaining $N \times (N-1)$ matrix is an $OA(N, 2^{N-1})$. (saturated)
  - the construction of $OA(N, 2^{N-1})$ is equivalent to the construction of $H_N$.
  - the $OA(N, 2^{N-1})$ can be used as a design matrix of $N-1$ 2-level factors while the $H_N$ can be used as a model matrix (for the main effect model), whose first column corresponds to the intercept term




$a+b=c+d \rightarrow a+b-c-d=0$
$a+c=b+d \rightarrow a-b+c-d=0$
$a+d=c+b \rightarrow a-b-c+d=0$

$\begin{bmatrix} a=d \\ a=c \\ a=b \end{bmatrix} \Rightarrow a=b=c=d$



# Hadamard Matrix

$4k$ ← Hadamard conjecture

N=4 → 8 → 16 → 32 → 64
N= 12 → 24 → 36 → 48 ..
N= 20 → 40 → 60 →

- An integer $N$ is a *Hadamard number* if an $H_N$ exists.
- For any Hadamard number $N$, $H_{2N}$ can be constructed as follows,

$H_{2N}^T H_{2N} = \begin{pmatrix} H_N^T & H_N^T \\ H_N^T & -H_N^T \end{pmatrix} \begin{pmatrix} H_N & H_N \\ H_N & -H_N \end{pmatrix}$

$= \begin{bmatrix} H_N^T H_N + H_N^T H_N & 0 \\ 0 & 2NI = H_N^T H_N + H_N^T H_N \end{bmatrix}$

$$H_{2N} = \begin{pmatrix} H_{N_{N\times N}} & H_N \\ H_N & -H_N \end{pmatrix}_{2N\times 2N}.$$

doubling: double run size & double # of factors.

- It is straightforward to verify that the $H_{2N}$ matrix is indeed a Hadamard matrix.
- By removing its first column, we then obtain an $OA(2N, 2^{2N-1})$ for any Hadamard number $N$.

Note: there could be more nonisomorphic OAs if # of factors < N-1, & some of them are not projection of Hadamard matrix.

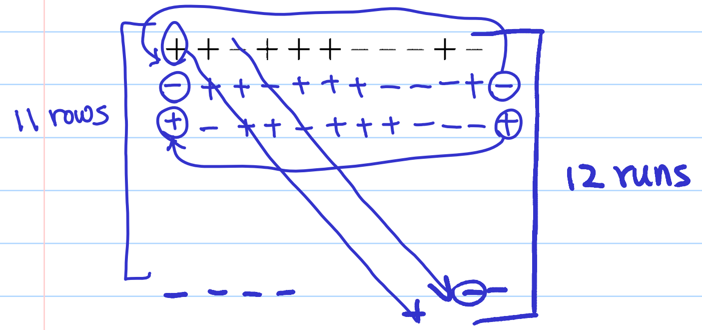
- Up to isomorphism, there is a unique Hadamard matrix for $N = 1, 2, 4, 8$, and 12. There are 5 nonisomrphic matrices of $N = 16$, 3 of $N = 20$, 60 of $N = 24$, and 487 of $N = 28$.

special case of Hadamard matrix

# Plackett-Burman Design (PBD)

saturated

- A large collection of $OA(N, 2^{N-1})$ was given to in Plackett and Burman (1946) for $N = 4k$ but not a power of two. → N=12, 20, 24, 28, ~~32~~, 36, ...
- These designs can be generated from the first row by *cyclic* arrangement.
  - the next row is generated from the ~~first~~ previous row by moving the entries of the previous row one position to the right and placing the last entry in the first position
  - the process continues until $N - 1$ rows are generated
  - a row of $-1$'s is then added as the last row, completing the design.



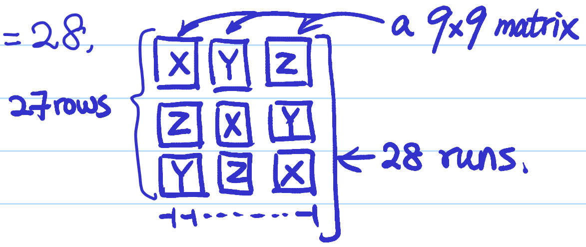



# Plackett-Burman Design (PBD)

Table 3: Generating Row Vectors for Plackett-Bruman Designs of Run Size $N$

| $N$ | Vector |
|---|---|
| 12 | + + − + + + − − − + − ← the vector appeared in LNp.4-11 |
| 20 | + + − − + + + + − + − + − − − − + + − |
| 24 | + + + + + − + − + + − − + + − − + − + − − − − |
| 36 | − + − + + + − − − + + + + + − + + + − − + − − − − + − + − + + − − + − |
| 44 | + + − − + − + − − + + + − + + + + + − − − + − + + + − − − − − + − − − |
| | + + − + − + + − |

N=28,



# Projection Properties of PBDs

Also, if exp't only has few factors, use which columns for them? reuse of d.f. → analysis strategy ← $OA(N, 2^{N-1})$

① do factor screening to identify important "factors"

② project design matrix to the important factors → smaller DM.

③ Use the smaller DM to fit a model including all ME's + all/some interactions.

- Why needs to study projection property?
  - Take the 12-run PBD as an example.
  - Suppose the analysis of main-effect-only model suggests that there are only three significant main effects.
  - ∵ complex aliasing. After projected onto the three factors, can the desgin matrix allow interactions among the three factors to be estimated?
  - The estimability of interactions after projection is referred to as **hidden projection** property. c.f. geometric projection property
- For the 12-run PBD — can fit all (3) MEs & all interactions
  - Any set of three columns of the design consists of the sum of a $2^3$ design and a $2^{3-1}$ design with $\mathbf{I} = -ABC$
  - Any projection onto four columns of the design is isomorphic to the one given in Figure 1. 12 distinct runs  $12-1-5=6$
  - Any projection onto five factors allows all the main effects and five or six two-factor interactions among the five factors to be estimated.



# Projection Properties of PBDs

① need 5 additional runs → $2^4$

② need $(-,+,-,-)$ → $2^{4-1}_{IV}$, $I = ABCD$

③ can estimate any (all MEs + $(11-1-4=6)$ 2fi's)

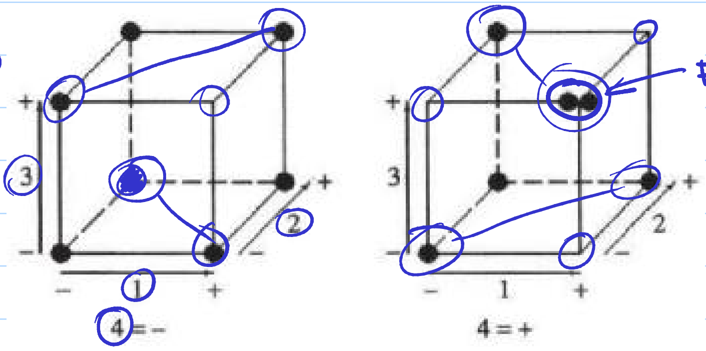



Figure 1: Projection of $OA(12, 2^{11})$ in Appendix 8A.1 (textbook) onto its first four columns

- **Theorem.** (Cheng, 1995) Let $\mathbf{X}$ be any $OA(N, 2^k)$ ($k \leq N-1$) with $k \geq 4$ and $N$ not a multiple of 8 ($N = 12, 20, 28, \ldots$). Then the projection of $\mathbf{X}$ onto any four factors has the property that all the main effects and two-factor interactions (# of 2fi = 6) among the four factors are estimable when the higher-order interactions are negligible.
- More details and results on the hidden projection properties can be found in Lin and Draper (1992) and Wang and Wu (1995)

# Hall's Designs

1 regular
4 nonregular

- Hall (1961) proved that there are exactly five nonisomorphic Hadamard matrices for $N = 16$ ($\Rightarrow$ five nonisomorphic $OA(16, 2^{15})$) ← saturated.
- The saturated $\boxed{2_{III}^{15-11}}$ design is a regular $OA(16, 2^{15})$ and is called a type I design
  A, B, C, D, AB, AC, ⋯, CD, ABC, ⋯, BCD, ABCD
  E F O
- The other four nonregular $OA(16, 2^{15})$s are given in Appendix 8B (textbook), in the ascending order of their "nonregularity"
  - The type II design is the closest to being regular in that it has the smallest number of pairs of main effect and two-factor interactions that are partially aliased
    A B ⋮ O   ME ↔ 2fi   AB AC ⋮ NO
  - On the other hand, the type V design is the most nonregular
- The four nonregular $OA(16, 2^{15})$s may be considered as alternatives to the regular one, expecially if the complex aliasing of the former deisngs is considered to be an advantage

❖ **Reading**: textbook, 8.4



# Useful Orthogonal Arrays

- Collection in Appendix 8A and 8C of textbook:

  d.f. 12-1-2-4=5 left
  cannot be increased
  d.f. 18-1-1-14 = 2 left

  | | | |
  |---|---|---|
  | $*OA(12, 2^{11})$ PB | $OA(\boxed{12}, 3^1 2^4)$ | $*OA(18, \boxed{2}^1 3^7)$ |
  | $*OA(18, 6^1 3^6)$ | $OA(20, 2^{19})$ PB | $OA(24, 2^{23})$ PB |
  | $OA(24, \boxed{3}^1 2^{16})$ | $OA(24, 6^1 2^{14})$ | $*OA(\boxed{36}, 2^{11} 3^{12})$ |
  | $OA(36, 3^7 6^3)$ | $OA(36, 2^8 6^3)$ | $OA(48, 2^{11} 4^{12})$ |
  | $OA(50, 2^1 5^{11})$ | $OA(54, 2^1 3^{25})$ | |

  d.f. 24-1-2-16 = 5
  d.f. 36-1-11-24=0

  * especially useful

- Learn to choose and use the design tables in the collection.
- A library of orthogonal arrays: http://neilsloane.com/oadir/

method of replacement

# $OA(18, 2^1 3^7)$ and $OA(18, 6^1 3^6)$

MEs & 2fi's of the 2 columns ⊥ to columns 3-8.

Recall: $2^{n-p}$ FFD in $2^k$ blocks

Table 4: $OA(18, 2^1 3^7)$ (columns 1–8) and $OA(18, 6^1 3^6)$ (columns 1′ and 3–8)

| Run | 1′ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 2 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 |
| 3 | 0 | 0 | 0 | 2 | 2 | 2 | 2 | 2 | 2 |
| 4 | 1 | 0 | 1 | 0 | 0 | 1 | 1 | 2 | 2 |
| 5 | 1 | 0 | 1 | 1 | 1 | 2 | 2 | 0 | 0 |
| 6 | 1 | 0 | 1 | 2 | 2 | 0 | 0 | 1 | 1 |
| 7 | 2 | 0 | 2 | 0 | 1 | 0 | 2 | 1 | 2 |
| 8 | 2 | 0 | 2 | 1 | 2 | 1 | 0 | 2 | 0 |
| 9 | 2 | 0 | 2 | 2 | 0 | 2 | 1 | 0 | 1 |
| 10 | 3 | 1 | 0 | 0 | 2 | 2 | 1 | 1 | 0 |
| 11 | 3 | 1 | 0 | 1 | 0 | 0 | 2 | 2 | 1 |
| 12 | 3 | 1 | 0 | 2 | 1 | 1 | 0 | 0 | 2 |
| 13 | 4 | 1 | 1 | 0 | 1 | 2 | 0 | 2 | 1 |
| 14 | 4 | 1 | 1 | 1 | 2 | 0 | 1 | 0 | 2 |
| 15 | 4 | 1 | 1 | 2 | 0 | 1 | 2 | 1 | 0 |
| 16 | 5 | 1 | 2 | 0 | 2 | 1 | 2 | 0 | 1 |
| 17 | 5 | 1 | 2 | 1 | 0 | 2 | 0 | 1 | 2 |
| 18 | 5 | 1 | 2 | 2 | 1 | 0 | 1 | 2 | 0 |



Cf.

$36 = 2^2 \times 3^2$

cross array:

$2^{3-1} \times 3^{\cancel{4}-1}$

# $OA(36, 2^{11} 3^{12})$

11 ↔ 3

12 ↔ 4

Table 5: $OA(36, 2^{11} 3^{12})$

| Run | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |
| 4 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 2 |
| 5 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 2 | 0 | 0 | 0 | 0 |
| 6 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 2 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 |
| 7 | 0 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 1 | 1 | 1 | 0 | 0 | 1 | 2 | 0 | 1 | 2 | 2 | 0 | 1 | 1 | 2 |
| 8 | 0 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 2 | 0 | 1 | 2 | 0 | 0 | 1 | 2 | 2 | 0 |
| 9 | 0 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 1 | 1 | 1 | 2 | 2 | 0 | 1 | 2 | 0 | 1 | 1 | 2 | 0 | 0 | 1 |
| 10 | 0 | 1 | 0 | 1 | 1 | 0 | 1 | 1 | 0 | 0 | 1 | 0 | 0 | 2 | 1 | 0 | 2 | 1 | 2 | 1 | 0 | 2 | 1 |
| 11 | 0 | 1 | 0 | 1 | 1 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 1 | 0 | 2 | 1 | 0 | 2 | 0 | 2 | 1 | 0 | 2 |
| 12 | 0 | 1 | 0 | 1 | 1 | 0 | 1 | 1 | 0 | 0 | 1 | 2 | 2 | 1 | 0 | 2 | 1 | 0 | 1 | 0 | 2 | 1 | 0 |
| 13 | 0 | 1 | 1 | 0 | 1 | 1 | 0 | 1 | 0 | 1 | 0 | 0 | 1 | 2 | 0 | 2 | 1 | 0 | 2 | 2 | 1 | 0 | 1 |
| 14 | 0 | 1 | 1 | 0 | 1 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 2 | 0 | 1 | 0 | 2 | 1 | 0 | 0 | 2 | 1 | 2 |
| 15 | 0 | 1 | 1 | 0 | 1 | 1 | 0 | 1 | 0 | 1 | 0 | 2 | 0 | 1 | 2 | 1 | 0 | 2 | 1 | 1 | 0 | 2 | 0 |
| 16 | 0 | 1 | 1 | 1 | 0 | 1 | 1 | 0 | 1 | 0 | 0 | 0 | 1 | 2 | 1 | 0 | 0 | 2 | 1 | 2 | 2 | 1 | 0 |
| 17 | 0 | 1 | 1 | 1 | 0 | 1 | 1 | 0 | 1 | 0 | 0 | 1 | 2 | 0 | 2 | 1 | 1 | 0 | 2 | 0 | 0 | 2 | 1 |
| 18 | 0 | 1 | 1 | 1 | 0 | 1 | 1 | 0 | 1 | 0 | 0 | 2 | 0 | 1 | 0 | 2 | 2 | 1 | 0 | 1 | 1 | 0 | 2 |

$$OA(36, 2^{11}3^{12})$$

Table 5: Continued

| Run | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 19 | 1 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 1 | 0 | 0 | 1 | 0 | 2 | 2 | 2 | 0 | 1 | 1 | 0 | 1 | 2 |
| 20 | 1 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 1 | 0 | 1 | 2 | 1 | 0 | 0 | 0 | 1 | 2 | 2 | 1 | 2 | 0 |
| 21 | 1 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 1 | 0 | 2 | 0 | 2 | 1 | 1 | 1 | 2 | 0 | 0 | 2 | 0 | 1 |
| 22 | 1 | 0 | 1 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 1 | 0 | 1 | 1 | 2 | 2 | 0 | 1 | 0 | 0 | 2 | 2 | 1 |
| 23 | 1 | 0 | 1 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 1 | 1 | 2 | 2 | 0 | 0 | 1 | 2 | 1 | 1 | 0 | 0 | 2 |
| 24 | 1 | 0 | 1 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 1 | 2 | 0 | 0 | 1 | 1 | 2 | 0 | 2 | 2 | 1 | 1 | 0 |
| 25 | 1 | 0 | 0 | 1 | 1 | 1 | 0 | 1 | 1 | 0 | 0 | 0 | 2 | 1 | 0 | 1 | 2 | 2 | 0 | 2 | 0 | 1 | 1 |
| 26 | 1 | 0 | 0 | 1 | 1 | 1 | 0 | 1 | 1 | 0 | 0 | 1 | 0 | 2 | 1 | 2 | 0 | 0 | 1 | 0 | 1 | 2 | 2 |
| 27 | 1 | 0 | 0 | 1 | 1 | 1 | 0 | 1 | 1 | 0 | 0 | 2 | 1 | 0 | 2 | 0 | 1 | 1 | 2 | 1 | 2 | 0 | 0 |
| 28 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 1 | 0 | 2 | 1 | 1 | 1 | 0 | 0 | 2 | 1 | 2 | 0 | 2 |
| 29 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 1 | 1 | 0 | 2 | 2 | 2 | 1 | 1 | 0 | 2 | 0 | 1 | 0 |
| 30 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 1 | 2 | 1 | 0 | 0 | 0 | 2 | 2 | 1 | 0 | 1 | 2 | 1 |
| 31 | 1 | 1 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 1 | 1 | 0 | 2 | 2 | 2 | 1 | 2 | 1 | 1 | 0 | 1 | 0 | 0 |
| 32 | 1 | 1 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 2 | 0 | 2 | 2 | 1 | 2 | 1 | 1 |
| 33 | 1 | 1 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 1 | 1 | 2 | 1 | 1 | 1 | 0 | 1 | 0 | 0 | 2 | 0 | 2 | 2 |
| 34 | 1 | 1 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 1 | 0 | 0 | 2 | 0 | 1 | 2 | 1 | 2 | 0 | 1 | 1 | 2 | 0 |
| 35 | 1 | 1 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 1 | 0 | 1 | 0 | 1 | 2 | 0 | 2 | 0 | 1 | 2 | 2 | 0 | 1 |
| 36 | 1 | 1 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 1 | 0 | 2 | 1 | 2 | 0 | 1 | 0 | 1 | 2 | 0 | 0 | 1 | 2 |

2 levels ← → 3 levels

❖ **Reading**: textbook, 8.5