Nonregular Designs

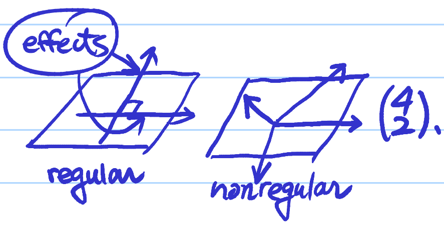
I=ABCD
A=BCD , AB=CD
B=ACD ,
D=ABC . AD=BC

e.g. 2^{n-p}, 3^{n-p} FFD.

- Regular designs: designs whose design matrices are constructed through defining relations among factors
  - iff — For regular designs, any two factorial effects (in model matrix of all possible effects.) either can be estimated (orthogonal components in 3^{n-p}) independent of each orther or are fully aliased ↔ partial aliasing
  - orthogonal — $cor(eff_1, eff_2) \in \{+1, -1, 0\}$
- Nonregular designs: designs that are not regular designs $Cor(eff_1, eff_2) \in [-1, 1]$.
- In Tables 1 and 2, the design used does not belong to the 2^{k-p} series (Chapter 5) or the 3^{k-p} series (Chapter 6), because the latter would require run size as a power of 2 or 3. These designs belong to the class of orthogonal arrays.

disadvantage: complex aliasing
advantage: more estimable models

2x2



Example : $OA(12, 2^{11})$

y~A+B
y~A+B+AB
y~A+AB
y~A+C
important factors are A & B.
y~B+C

usually can use smaller run size

objective: factor screening ↔ effect screeing.

Table 1: Design Matrix and Lifetime Data, Cast Fatigue Experiment

run size −1.

Run	Factor A	B	C	D	E	F	G	8	9	10	11	Logged Lifetime
1	+	+	−	+	+	+	−	−	−	+	−	6.058
2	+	−	+	+	+	−	−	−	+	−	+	4.733
3	−	+	+	+	−	−	−	+	−	+	+	4.625
4	+	+	+	−	−	−	+	−	+	+	−	5.899
5	+	+	−	−	−	+	−	+	+	−	+	7.000
6	+	−	−	−	+	−	+	+	−	+	+	5.752
7	−	−	−	+	−	+	+	−	+	+	+	5.682
8	−	−	+	−	+	+	−	+	+	+	−	6.607
9	−	+	−	+	+	−	+	+	+	−	−	5.818
10	+	−	+	+	−	+	+	+	−	−	−	5.917
11	−	+	+	−	+	+	+	−	−	−	+	5.863
12	−	−	−	−	−	−	−	−	−	−	−	4.809

not a power of 2

Example : $OA(18, 2^1 3^7)$

different # of levels

Table 2: Design Matrix and Response Data, Blood Glucose Experiment

2 levels 3 levels

Run	Factor A	G	B	C	D	E	F	H	Mean Reading
1	0	0	0	0	0	0	0	0	97.94
2	0	0	1	1	1	1	1	1	83.40
3	0	0	2	2	2	2	2	2	95.88
4	0	1	0	0	1	1	2	2	88.86
5	0	1	1	1	2	2	0	0	106.58
6	0	1	2	2	0	0	1	1	89.57
7	0	2	0	1	0	2	1	2	91.98
8	0	2	1	2	1	0	2	0	98.41
9	0	2	2	0	2	1	0	1	87.56
10	1	0	0	2	2	1	1	0	88.11
11	1	0	1	0	0	2	2	1	83.81
12	1	0	2	1	1	0	0	2	98.27
13	1	1	0	1	2	0	2	1	115.52
14	1	1	1	2	0	1	0	2	94.89
15	1	1	2	0	1	2	1	0	94.70
16	1	2	0	2	1	2	0	1	121.62
17	1	2	1	0	2	0	1	2	93.86
18	1	2	2	1	0	1	2	0	96.10

not a power of 2
nor a power of 3

Orthogonal Arrays

of distinct # of levels

run size
# of factors

- An **orthogonal array** $OA(N, s_1^{m_1} \ldots s_\gamma^{m_\gamma}, t)$ of strength t is an $N \times m$ matrix, $m = m_1 + \ldots + m_\gamma$, in which m_i columns have s_i (≥ 2) symbols or levels such that, for any t columns, all possible combinations of symbols appear equally often in the matrix. ← projection interpretation → projected onto any t columns → form full factorial
  - A regular design is an OA of strength $R - 1$, where R is the resolution of the regular design
    $t = R - 1 \Leftrightarrow t + 1 = R$
  - $\gamma = 1 \Rightarrow$ **symmetrical** OAs; $\gamma > 1 \Rightarrow$ **asymmetric** (or **mixed-level**) OAs
- For OA of strength two, the index $t = 2$ is dropped for simplicity.
- An $OA(12, 2^{11})$ is used in Table 1 and an $OA(18, 2^1 3^7)$ is used in Table 2.

OA of strength 2 → MEs are mutually orthogonal (mo)
MEs & 2fi's — orthogonal / fully aliased / partially aliased
OA of strength 3 → ME's mo
ME's ↔ 2fi, mo
2fi ↔ 2fi
OA of strength 4 → {MEs, 2fi's} mo.

❖ **Reading**: textbook, 8.1, 8.3

Why Using non-regular Orthogonal Array

regular designs

at most 7 factors.

run size: 4, 8, 16, 32, 64, ...

- **Run size economy.** Suppose 8-11 factors at two levels are to be studied. Using an $OA(\underline{12}, 2^{11})$ will save 4 runs over a 16-run $\boxed{2^{k-p}}$ design. Similarly, suppose 5-7 factors at three levels are to be studied. Using an $OA(\underline{18}, 3^7)$ will save 9 runs over a 27-run $\boxed{3^{k-p}}$ design. run size: 9, 27, 81, ...

at most 4 factors.

- **Flexibility.** Many OAs exist for flexible combinations of factor levels (see the collection of some useful OAs on later slides).

∵ complex aliasing ⇒ analysis is more complex.

- Analysis strategy for experiments based on OA can be found in Chapter 9 of WH.

Cf

regular design: if a model is estimable, all the effects in the model are mutually orthogonal. But, fewer estimable models

non-regular design: the effects in an estimable model may not be mutually orthogonal, But, more estimable models.

❖ **Reading**: textbook, 8.2

A Lemma on Orthogonal Arrays

- **Lemma.** For an orthogonal array $OA(N, s_1{}^{m_1} \cdots s_\gamma{}^{m_\gamma}, t)$, its run size N must be divisible by the least common multiple of $\prod_{i=1}^{\gamma} s_i{}^{k_i}$, for all possible combinations of k_i with $k_i \le m_i$ and $\sum_{i=1}^{\gamma} k_i = t$, $i = 1, \ldots, \gamma$.

  Proof:

  Project the OA onto any t factors ⇒ form a full factorial with equal replicates

  According to the definition of OAs, in any t columns of the array with k_i columns having s_i symbols, $k_i \le m_i$, $\sum_{i=1}^{\gamma} k_i = t$, each of the $\prod_{i=1}^{\gamma} s_i{}^{k_i}$ combinations of symbols appears equally often.

  $\Rightarrow N$ is divisible by the least common multiple of $\prod_{i=1}^{\gamma} s_i{}^{k_i}$ over these choices of t columns.

  ⇒ run size N divisible by

- Examples:

  OA of 2-level factors: run size N=4k (k∈Z+), N=4, 8, 12, 16, 20, 24, ...

  OA: 3-level factors: run size N=9k, N=9, 18, 27, 36, 45, ...

  $OA(N, 2^2 3^2)$, N must be divisible by 4, 6, 9, LCM=36.

  $OA(N, 2^1 3^3)$, N : : : : 6, 9, LCM=18

  $OA(N, 2^3 3^1)$, N : : : : 4, 6, LCM=12

❖ **Reading**: textbook, 8.3

isomorphic design matrices have same statistical property

Isomorphic Designs

同構 ← same exp't.

- Two $\boxed{\textit{design matrices}}$ are said to be **isomorphic** (or **equivalent**) if one design matrix can be obtained from the other by
  - row permutations,
  - column permutations,
  - relabeling of levels

$M^{(1)}_{ij} \neq M^{(2)}_{ij}$ if $\exists\, ij$

Why?

sign switch for 2-level case.

# of level ≥ 3 for qualitative factors