Box-Behnken Designs - Some Properties

- Only require three levels of each factor (c.f., CCDs require 5 levels when $\alpha \neq 1$)
- All the design points (except the center) have length two, i.e., they all lie on the same sphere
  - These designs are particularly usited for spherical regions
  - Because of the spherical property, there should be at least three to five runs at the center point
- The design for $k = 4$ is rotatable and the other designs are nearly rotatable
- Orthogonal blocking

Box-Behnken Designs - Orthogonal Blocking

- For $k = 4, 5$, the designs can be arranged in orthogonal blocks
- $k = 4$, can be arranged in 3 blocks

$$\begin{bmatrix} \pm1 & \pm1 & 0 & 0 \\ 0 & 0 & \pm1 & \pm1 \\ 0 & 0 & 0 & 0 \\ \hline \pm1 & 0 & 0 & \pm1 \\ 0 & \pm1 & \pm1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline \pm1 & 0 & \pm1 & 0 \\ 0 & \pm1 & 0 & \pm1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- $k = 5$, can be arranged in 2 blocks

$$\begin{bmatrix} \pm1 & \pm1 & 0 & 0 & 0 \\ 0 & 0 & \pm1 & \pm1 & 0 \\ 0 & \pm1 & 0 & 0 & \pm1 \\ \pm1 & 0 & \pm1 & 0 & 0 \\ 0 & 0 & 0 & \pm1 & \pm1 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & \pm1 & \pm1 & 0 & 0 \\ \pm1 & 0 & 0 & \pm1 & 0 \\ 0 & 0 & \pm1 & 0 & \pm1 \\ \pm1 & 0 & 0 & 0 & \pm1 \\ 0 & \pm1 & 0 & \pm1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Uniform Shell Designs

- The designs have k^2+k+1 points, which consist of
  - k^2+k points uniformly spaced on the k-dimensional sphere, and
  - one center point
- For $k=2$, it can be described by a regular hexagon plus the center point

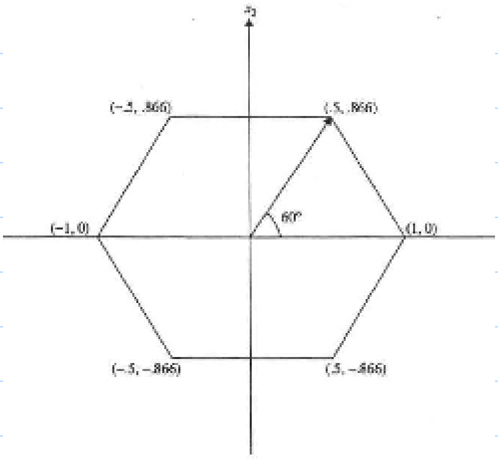

Figure 10: Uniform shell design, k=2

- For $k=3$, it is identical to the Box-Behnken design

Uniform Shell Designs

- Uniformity property
  - Covering region of a design point: collection of points on the sphere that are closer to this design point than to the other design points.
  - Uniformity ensures that the covering region of each design point has the *same* spherical area.
- Run size comparison
  - For $k>5$, its run size is much larger than $(k+1)(k+2)/2$
- For $2\le k\le 5$, the designs are given in Table 10C.1 (textbook)
  - These designs can be rotated with affecting the uniformity property. However, rotation will likely increase the number of levels.
  - In the table, the design points minimize the number of the levels
- A disadvantage of spherical designs: lack of coverage of the region between the origin and the spherical shell

v **Reading**: textbook, 10.8

Analysis Strategies for Multiple Responses

- Multople response analysis starts with building a regression fitted model for each response separately
- Some analysis strategies
  - Overlaid contour plots (if there are few responses and only two or three important input variables.)
  - Constrained optimization (if, among the responses, one is of primary importance)
  - Desirability function (if the goal is to find an optimal balance among several response characteristics)
- The desirability function approach transform each predicted response $\hat{y}_i$ to a desirability value d_i, where $0 \le d_i \le 1$
  - The value of the **desirability function** d increases as the "desirability" of the corresponding response increases.

Analysis Strategies for Multiple Responses

- Desirability function
  - For nominal-the-best problem:

$$d = \begin{cases} \left|\dfrac{\hat{y}-L}{t-L}\right|^{\alpha_1}, & L \le \hat{y} \le t \\ \left|\dfrac{\hat{y}-U}{t-U}\right|^{\alpha_2}, & t \le \hat{y} \le U \\ 0, & \text{otherwise} \end{cases}$$

  - For smaller-the-better problem

$$d = \begin{cases} \left|\frac{\hat{y}-U}{a-U}\right|^{\alpha}, & a \le \hat{y} \le U \\ 0, & \hat{y} > U \end{cases}$$

  - For larger-the-better problem

$$d = \begin{cases} \left|\frac{\hat{y}-L}{U-L}\right|^{\alpha}, & L \le \hat{y} \le U \\ 1, & \hat{y} > U \\ 0, & \hat{y} < L \end{cases}$$

Analysis Strategies for Multiple Responses

- Desirability function II
  - In some practical situations, the L and U values in the desirability function I cannot be properly chosen.
  - For nominal-the-best problem:

$$d = \begin{cases} \exp\{-c_1|\hat{y}-t|^{\alpha_1}\}, & -\infty < \hat{y} \le t \\ \exp\{-c_2|\hat{y}-t|^{\alpha_2}\}, & t \le \hat{y} < \infty \end{cases}.$$

  - For smaller-the-better problem

$$d = \exp\{-c|\hat{y}-a|^{\alpha}\},\ a \le \hat{y} < \infty.$$

  - For larger-the-better problem

$$d = \begin{cases} 1-\exp\{-c\hat{y}^{\alpha}\}/\exp\{-cL^{\alpha}\}, & L \le \hat{y} < \infty \\ 0, & \hat{y} < L \end{cases}$$

Analysis Strategies for Multiple Responses

- Overall desirability function D
  - Weighted geometric mean

$$D = d_1^{w_1} d_2^{w_2} \cdots d_m^{w_m},$$

    where the weights w_i's satisfy $0 < w_i < 1$ and $w_1 + \cdots + w_m = 1$.
    * $\ln D = \sum_{i=1}^{m} w_i \ln d_i$
    * If any $d_i = 0$, then $D = 0$
    * When $w_1 = \cdots = w_m = 1/m$, D is the geometric mean
  - Weighted arithmetic mean

$$D = w_1 d_1 + w_2 d_2 + \cdots + w_m d_m$$

- Any setting for the input factors that maximizes the D value is chosen to be one which achieves an optimal balance over the m responses.
  - After finding such a setting, it should be verified whether all constraints on the responses are satisfied

v **Reading**: textbook, 10.6