Response Surface Methodology (RSM)

- Experimental study focusing on the relationship between the response and the input factors.
- Common purposes in RSM:
  - Optimizing the response
  - Mapping the response surface over a region of interest
- Sequential experimentation strategy for optimization
- RSM is an effective tool especially when
  - input factors are quantitative,
  - there are only a few of input factors.

Ranitidine Experiment

- Consider an experiment to study three quantitative factors with up to 5 levels.

Table 1: Factors and Levels, Ranitidine Experiment

Factor	Levels
A. pH	2, 3.42, 5.5, 7.58, 9
B. voltage (kV)	9.9, 14, 20, 26, 30.1
C. α-CD (mM)	0, 2, 5, 8, 10

- The design matrix and the data are given on the next page. The design differs from 2^{k-p} design in two respects :
  - 6 replicates at the center,
  - 6 runs along the three axes.

It belongs to the class of *central composite designs* (CCD).

Ranitidine Experiment

Table 2: Design Matrix and Response Data

Run	Factor A	B	C	CEF	ln CEF
1	−1	−1	−1	17.293	2.850
2	1	−1	−1	45.488	3.817
3	−1	1	−1	10.311	2.333
4	1	1	−1	11757.084	9.372
5	−1	−1	1	16.942	2.830
6	1	−1	1	25.400	3.235
7	−1	1	1	31697.199	10.364
8	1	1	1	12039.201	9.396
9	0	0	−1.67	7.474	2.011
10	0	0	1.67	6.312	1.842
11	0	−1.68	0	11.145	2.411
12	0	1.68	0	6.664	1.897
13	−1.68	0	0	16548.749	9.714
14	1.68	0	0	26351.811	10.179
15	0	0	0	9.854	2.288
16	0	0	0	9.606	2.262
17	0	0	0	8.863	2.182
18	0	0	0	8.783	2.173
19	0	0	0	8.013	2.081
20	0	0	0	8.059	2.087

Ranitidine Experiment

The CCD has three parts

(1) *cube* (or corner) points, (2) *axial* (or star) points, (3) *center* points.

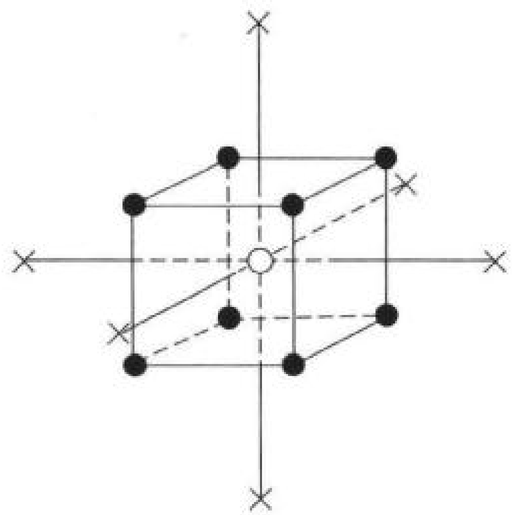


Figure 1: A Central Composite Design in Three Dimensions [cube point (dot), star point (cross), center point (circle)].

- **Reading**: textbook, 10.1

Sequential Nature of RSM

- Basic concepts
  - For a process/system involving a response y and *input factors* $X_1, \ldots, X_k$, model their relationship by

    $$y = f(X_1, \ldots, X_k) + \varepsilon,$$

    where f is unknown.
  - *Locally* approximate f by

    $$f(X_1, \ldots, X_k) \approx \sum_{i=1}^{m} \beta_i \cdot g_i(X_1, \ldots, X_k),$$

    where β_i's: unknown parameters, and g_i's: known functions of $X_1, \ldots, X_k$
  - Conduct experiments and collect data to estimate β_i's
  - Use *fitted response surface* $\hat{f}$ to understand f:

    $$\hat{f} = \sum_{i=1}^{m} \hat{\beta}_i \cdot g_i(X_1, \ldots, X_k) \longrightarrow f$$

Sequential Nature of RSM

- Sequential experimentation to maximize/minimize the response or to achieve a desirec value of the response:

  1. **Screening Experiment :** When many variables are considered, some are likely to be inert. Use *a* 2^{k-p} *design or an OA.*

  2. Once a small number of important factors is identified, if the experimental region is far from the optimum (examined by **curvature check**), use the **first-order model**

     $$y = \beta_0 + \sum_{i=1}^{k} \beta_i x_i + \varepsilon, \quad (1)$$

     to fit the data.

Sequential Nature of RSM

3. Based on the fitted first-order model, find the steepest ascent direction and perform a search along this direction (called **steepest ascent search**).
   – Steps 2 and 3 may be repeated until reaching the optimum region (e.g. peak of the surface).
   – Alternative to steps 2 and 3: **rectangular grid search**

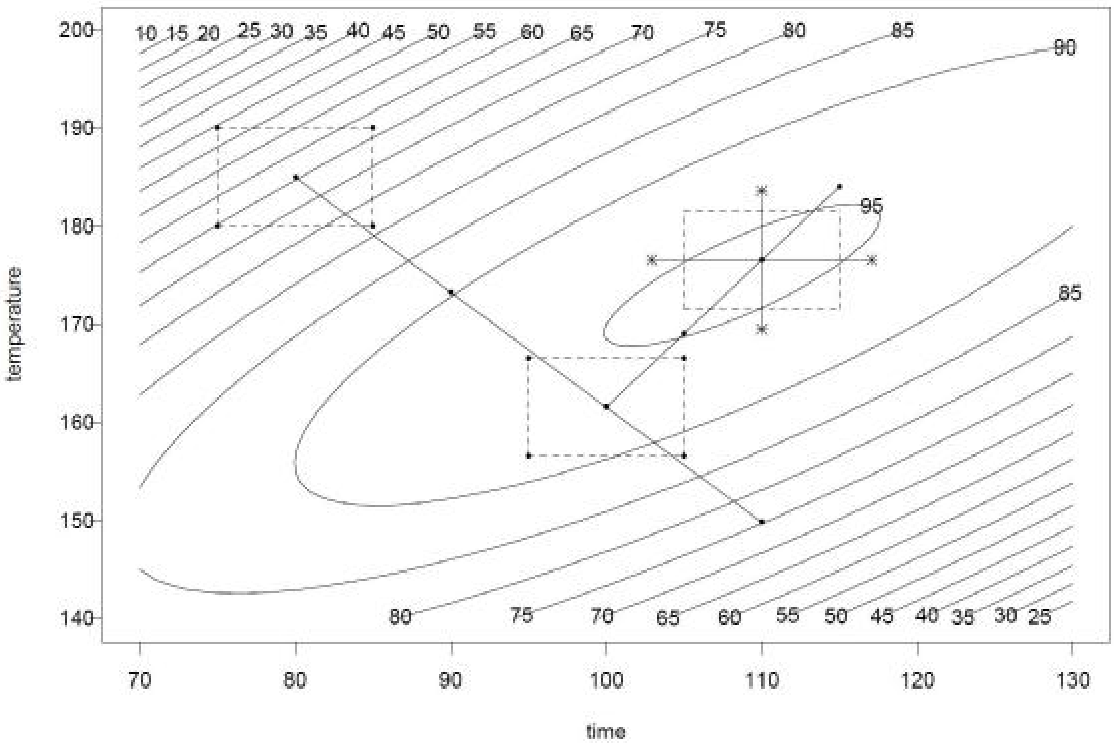
4. To capture the curvature effects, use a **second-order design** (like the central composite design). Fit a **second-order model**

$$y = \beta_0 + \sum_{i=1}^{k} \beta_i x_i + \sum_{i<j}^{k} \beta_{ij} x_i x_j + \sum_{i=1}^{k} \beta_{ii} x_i^2 + \varepsilon, \quad \varepsilon \sim N(0, \sigma^2 I) \qquad (2)$$

to data. Use the fitted model (with insignificant terms dropped) to do *contour plots* and find the *optimum* conditions.

A graphical illustration of these steps is given on next page.

Sequential Exploration of Response Surface



v **Reading**: textbook, 10.2

Center Points and Curvature Check

- Purposes of adding *replicated* center points to a 2-level first-order experiment
  - it allows the check of the overall curvature effect,
  - it provides an unbiased estimate of the process error variance
  - it allows the over-fitting/under-fitting check in residual plot

- Suppose that
  - the first-order experiment is based on a 2-level orthogonal design with run size n_f, and
  - n_c center point runs are added to the first-order experiment.

Center Points and Curvature Check

- Let $\bar{y}_f$ = sample average over factorial runs with level ± 1,
  $\bar{y}_c$ = sample average at n_c center points with level 0
- Under the second-order model

$$y = \beta_0 + \sum_{i=1}^{k} \beta_i x_i + \sum_{i<j}^{k} \beta_{ij} x_i x_j + \sum_{i=1}^{k} \beta_{ii} x_i^2 + \varepsilon, \quad \varepsilon \sim N(0, \sigma^2 I)$$

  - $E(\bar{y}_c) = \beta_0$,
  - $E(\bar{y}_f) = \beta_0 + \sum_{i=1}^{k} \beta_{ii}$,
  - $E(\bar{y}_f - \bar{y}_c) = \sum_{i=1}^{k} \beta_{ii}$, and
  - $Var(\bar{y}_f - \bar{y}_c) = \sigma^2(\frac{1}{n_f} + \frac{1}{n_c})$.
- *overall curvature* parameter: $\beta_{11} + \cdots + \beta_{kk}$
  - In using the overall curvature to measure the curvature effects, it is possible that some of the large β_{ii} cancel each other, thus yielding a very small $\sum_{i=1}^{k} \beta_{ii}$.
  - This rarely occurs unless exp'tal region near a saddle point of the f.

Center Points and Curvature Check

- To test the hypotheses

$$H_0 : \beta_{11} + \cdots + \beta_{kk} = 0 \ \text{ v.s. } \ H_a : \beta_{11} + \cdots + \beta_{kk} \neq 0$$

we can use $\bar{y}_f - \bar{y}_c$.

- Reject H_0 at level α if

$$\frac{|\bar{y}_f - \bar{y}_c|}{s\sqrt{\frac{1}{n_f} + \frac{1}{n_c}}} > t_{n_c-1, \frac{\alpha}{2}}$$

where s^2= sample variance based on n_c center runs is used to estimate σ^2.

- If the curvature check is not significant the search may continue with the use of another first-order exeriment and steepest ascent. Otherwise, it should be swithched to a second-order experiment.

Center Points and Curvature Check

- Justification from regression analysis:

v **Reading**: textbook, 10.3.1