# Half-normal Plot, Layer Growth Experiment

Similar to the coding of 2 2-level factors & their interaction → A B AB

the whole design matrix ($\underset{\sim}{C}$ & $\underset{\sim}{N}$) can be treated as $2^{(8+3)-4}_{IV}$ regular design with the same defining contrast subgroup given in LNp.2-14.

10/21

- Define

$$M_l = (M_1 + M_2) - (M_3 + M_4),$$
$$M_q = (M_1 + M_4) - (M_2 + M_3),$$
$$M_c = (M_1 + M_3) - (M_2 + M_4),$$

| M | | Me | Mg | $\boxed{\text{Mc}}$ = Me×Mg |
|---|---|---|---|---|
| 1 | coding → | +1 | + | + |
| 2 | | +1 | − | − |
| 3 | | − | − | + |
| 4 | | − | + | − |

benefit: orthogonality

model: $y \sim \sum \beta_i$ (all factorial effects formed by $\underset{\sim}{C}$ & $\underset{\sim}{N}$)$_i + \varepsilon$ ← no replicates

- From Figure 4, select $D$, $L$, $HL$ as the most significant effects.

  adjustment factor $\bar{z}$ ↑ ↑ can be used to reduce Var(y) significantly.

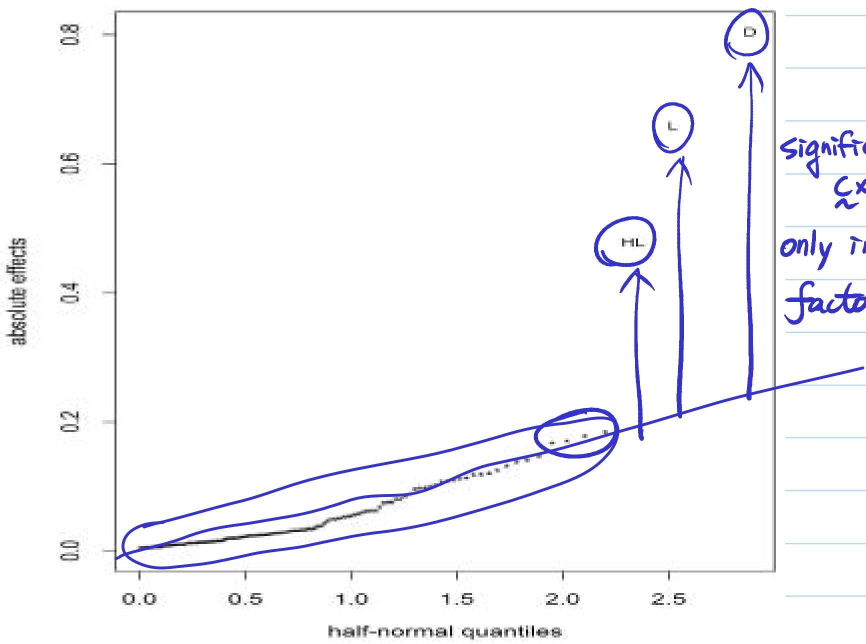
- How to deal with the next cluster of effects in Figure 4? Use **step-down multiple comparisons.**

  c.f. L causes larger variation reduction.

- After removing the top three points in Figure 4, make a half-normal plot (Figure 5) on the remaining points. The cluster of next four effects $M_l$, $H$, $CM_l$, $AHM_q$ appear to be significant.



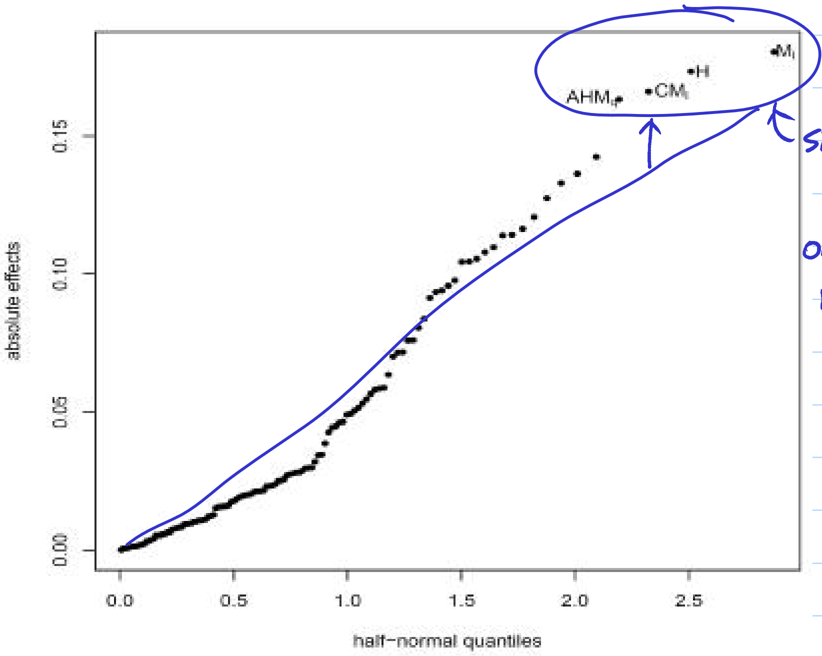
# Half-normal Plot of Factorial Effects




significant $\underset{\sim}{C} \times \underset{\sim}{N}$ interaction only involve noise factor L.

Figure 4: Half-Normal Plot of Response Model Effects, Layer Growth Experiment

# Second Half-normal Plot of Factorial Effects




significant C×N interaction
only involve noise factor M

Figure 5: Second Half-Normal Plot of Response Model Effects, Layer Growth Experiment



# Control-by-noise Interaction Plots

reduce larger variation

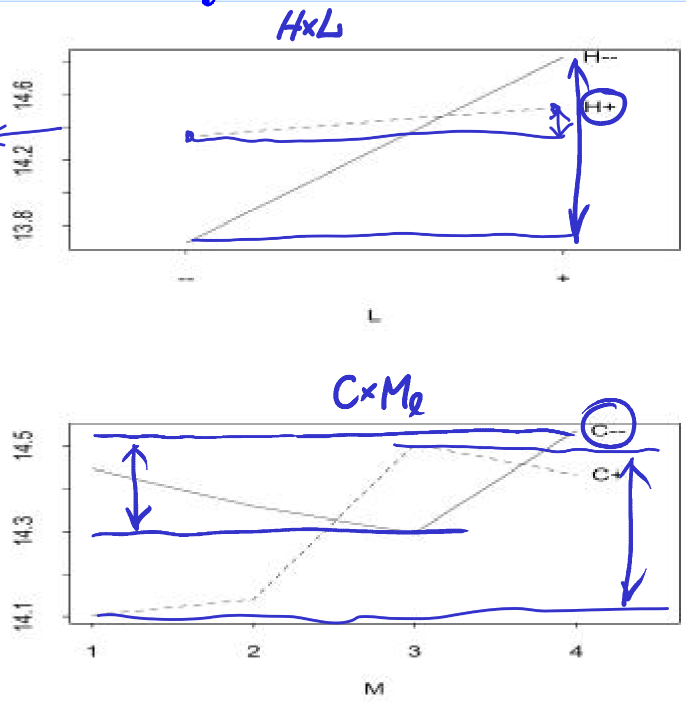


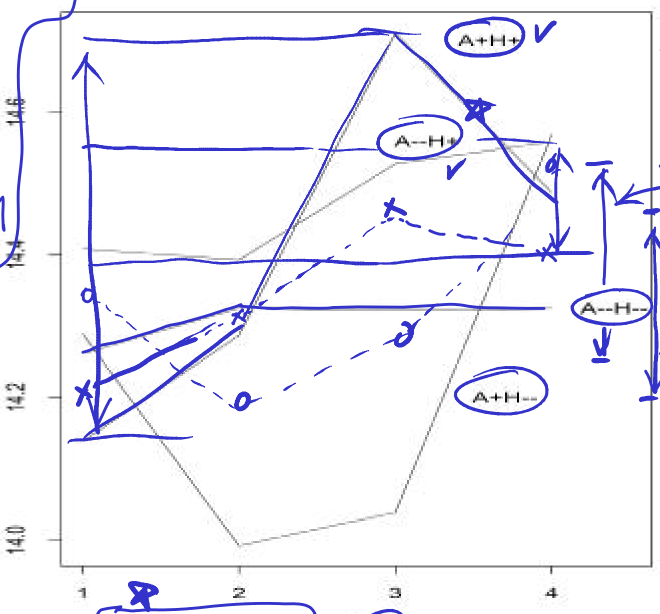

Figure 6: $H \times L$ and $C \times M$ Interaction Plots, Layer Growth Experiment

## $A \times H \times M$ Plot

regarded as one factor

$(A\times H)\times M$.

HL & AHM interaction offer physical interpretation of why variation caused by L & M related to A & M




recommendation: $\boxed{\text{H+ C- A-}}$ ← C.P. → location-dispersion modeling.

Figure 7: $A \times H \times M$ Interaction Plot, Layer Growth Experiment



# Response Modeling, Layer Growth Experiment

- The following model is obtained: adjust $E(y|\mathbf{s})$

$E(y|\mathbf{s},\mathbf{N})$

$$\hat{y} = 14.352 + 0.402 x_D + 0.087 x_H + 0.330 \boxed{x_L} - 0.090 x_{M_l}$$
$$-0.239 \boxed{x_H x_L} - 0.083 x_C x_{M_l} - 0.082 x_A x_H x_{M_q}. \quad (4)$$

- Recommendations: from C×N interaction plot.
  $H$: $-$ (position 2) to $\boxed{+}$ (position 6)
  $A$: $+$ (oscillating) to $-$ (continuous)
  $C$: $+$ (1210) to $-$ (1220)
  resulting in 37% reduction of thickness standard variation.

# Predicted Variance Model

- Assume $L$, $M_l$ and $M_q$ are random variables, taking $-1$ and $+1$ with equal probabilities. This leads to

indep.

L: 1 ← 1/2 +, 2 ← 1/2 −

M: 1/4 → 1, 1/4 → 2, 1/4 → 3, 1/4 → 4

| | Ml | Mq | Mc |
|---|---|---|---|
| 1 | + | + | + |
| 2 | + | − | − |
| 3 | − | − | + |
| 4 | − | + | − |

$$\begin{aligned} &x_L^2 = x_{M_l}^2 = x_{M_q}^2 = x_A^2 = x_C^2 = x_H^2 = 1, \\ &E(x_L) = E(x_{M_l}) = E(x_{M_q}) = 0, \\ &Cov(x_L, x_{M_l}) = Cov(x_L, x_{M_q}) = Cov(x_{M_l}, x_{M_q}) = 0. \end{aligned} \quad (5)$$

$Var(x_L) = Var(x_{M_l}) = Var(x_{M_q}) = 1 - 0 = 1.$

- From (4) and (5), we have

$Var_N(E(y|C,N))$

$$\begin{aligned} Var(\hat{y}) &= (.330 - .239x_H)^2 Var(x_L) + (-.090 - .083x_C)^2 Var(x_{M_l}) \\ &\quad + (.082x_A x_H)^2 Var(x_{M_q}) \\ &= \text{constant} + (.330 - .239x_H)^2 + (-.090 - .083x_C)^2 + (0.082)^2 x_A^2 x_H^2 \\ &= \text{constant} - 2(.330)(.239)x_H + 2(.090)(.083)x_C \\ &= \text{constant} - .158x_H + .015x_C. \end{aligned}$$

$E_N(E(y|C,N)) = 14.352 + 0.402x_? + 0.087x_H.$

Note: $x_{M_q}$'s ME is in the model, the term can be used to reduce variance.

- Choose $H+$ and $C-$. But factor $A$ is not present here. (Why? See explanation on p. 532).

**v Reading**: textbook, 11.5



Recall: C×N interactions → effects can be used to reduce variation

# Estimation Capacity for Cross Arrays

only consider regular design for simplicity

words formed only by noise factors; $2^{(3+3)-2}$; all C-type words × N-type words; words formed by control factors only

- Example 1. Control array is a $2_{III}^{3-1}$ design with $\mathbf{I} = \mathbf{ABC}$ and the noise array is a $2_{III}^{3-1}$ design with $\mathbf{I} = \mathbf{abc}$. The resulting cross array is a 16-run $2_{III}^{6-2}$ design with $\mathbf{I} = \mathbf{ABC} = \mathbf{abc} = \mathbf{ABCabc}$. Easy to show that all 9 control-by-noise interactions (2-factor) are clear, (but not the 6 main effects). This is indeed a general result stated next.

C×C 2fi ×, N×N 2fi ×

**Theorem:** Suppose a $2^{k-p}$ design $d_C$ is chosen for the control array, a $2^{m-q}$ design $d_N$ is chosen for the noise array, and a cross array, denoted by $d_C \otimes d_N$, is constructed from $d_C$ and $d_N$. ($2^{(m+k)-(p+q)}$)

estimability **(i)** If $\{\alpha_1, \ldots, \alpha_A\}$ ($\leftarrow$ # of effects $= A$) are the estimable factorial effects (among the control factors) in $d_C$ and $\{\beta_1, \ldots, \beta_B\}$ ($\leftarrow$ # of effects $= B$) are the estimable factorial effects (among the noise factors) in $d_N$, then $\{\alpha_i, \beta_j, \alpha_i\beta_j\}$ ($\leftarrow$ # of effects $= AB$) for $i = 1, \ldots, A$, $j = 1, \ldots, B$ are estimable in $d_C \otimes d_N$.

clearness **(ii)** All the $km$ control-by-noise interactions (i.e., two-factor interactions between a control factor main effect and a noise factor main effect) are $\boxed{\text{clear}}$ in $d_C \otimes d_N$.

☆ In general, C-type words $u_1, \ldots, u_{2^p-1}$ for $d_C \leftarrow 2^{k-p}$, resolution $\geq 3$
N-type words $v_1, \ldots, v_{2^q-1}$ for $d_N \leftarrow 2^{m-q}$, : $\geq 3$
the defining contrast subgroup for $d_C \times d_N \leftarrow$ is $2^{(k+m)-(p+q)}$

$$\{I, u_i, v_j, u_i v_j \mid i=1,\ldots,2^p-1;\ j=1,\ldots,2^q-1\} \quad \cdots (*)$$

# of words $= 1+(2^p-1)+(2^q-1)+(2^p-1)(2^q-1) = 2^{p+q}$

no need to restrict them as regular

☆ For Thm (LNp.2-27) (i): ⇑

model 1: $y \sim \beta_0 f_0(\underset{\sim}{C}) + \beta_1 f_1(\underset{\sim}{C}) + \cdots + \beta_{k-1} f_{k-1}(\underset{\sim}{C})$ : an estimable model for $d_C$

model 2: $y \sim \gamma_0 g_0(\underset{\sim}{N}) + \gamma_1 g_1(\underset{\sim}{N}) + \cdots + \gamma_{m-1} g_{m-1}(\underset{\sim}{N})$ : : : : $d_N$

then, $y \sim \sum_{0\le i\le k-1}\sum_{0\le j\le m-1} \alpha_{ij} \underbrace{f_i(\underset{\sim}{C}) \cdot g_j(\underset{\sim}{N})}_{\text{effects}}$ is an estimable model for $d_C \times d_N$

Why? ← Hint: interactions are defined by conditional main effects

☆ proof of Thm (LNp.2-27) (ii). ⇒ $\{\alpha_i, \beta_j, \alpha_i\beta_j\}$ is estimable.

$\alpha_i$ $\beta_j$ $\alpha_i\beta_j$

① $\alpha_i \leftrightarrow \alpha_{i'} \Rightarrow \alpha_i\alpha_{i'} \notin (*)$

② $\alpha_i \leftrightarrow \alpha_{i'}\beta_j \Rightarrow \alpha_i\alpha_{i'}\beta_j \notin (*)$



☆ proof of Thm (LNp.2-27) (ii),

- if a C×N 2fi appear in $\boxed{u_i v_j}$ (e.g. Aa appear in ABCabc) ← length $\geq 6$ ($\because$ length$(u_i) \geq 3$, $(v_j) \geq 3$)
  ⇒ C×N 2fi aliased with 4-fi or higher.
- for C-type words, say I=ABC
  $Aa = (\cancel{A}BC) \cdot \cancel{A}a = BCa$
- for N-type words, say I=abc → at most aliased with 3 fi.
  $Aa = (\cancel{a}bc) \cdot A\cancel{a} = Abc$

☆ Thm (LNp.2-27) can be generalized to $r^{k-p}$ design, r: prime #.

☆ Thm (LNp.2-27) (i) is still valid for $d_C, d_N$ not regular designs.

# Cross Arrays or Single Arrays?

construct design matrix for control factors & noise factors seperately

use all (control + noise) factors to construct a design matrix

can estimate all factorial effects, e.g., ABab, ACab, ...

$2^5$ = 32 runs

$d_{C\times}$ $d_N$

- Three control factors $A$, $B$, $C$ two noise factors $a$, $b$: $2^3 \times 2^2$ design, allowing all main effects and two-factor interactions to be clearly estimated.

  modeling: $y \sim$ all ME + all 2fi

- Use a single array with 16 runs for all five factors: a resolution V $2^{5-1}$ design with $\mathbf{I} = ABCab$ or $\mathbf{I} = -ABCab$, all main effects and two-factor interactions are clear. (See Table 7)

- Single arrays can have smaller runs, but cross arrays are easier to use and interpret.



# 32-run Cross Array and 16-run Single Arrays

I=ABCab

A B C a b=ABCa

$2^4$ full

cannot use location-dispersion analysis

not performed under the same settings of noise factors

$d_N$

Table 7: 32-Run Cross Array

| | | | a | + | + | − | − |
|---|---|---|---|---|---|---|---|
| | | | b | + | − | + | − |
| Runs | A | B | C | | | | |
| 1–4 | + | + | + | • | ○ | ○ | • |
| 5–8 | + | + | − | ○ | • | • | ○ |
| 9–12 | + | − | + | ○ | • | • | ○ |
| 13–16 | + | − | − | • | ○ | ○ | • |
| 17–20 | − | + | + | ○ | • | • | ○ |
| 21–24 | − | + | − | • | ○ | ○ | • |
| 25–28 | − | − | + | • | ○ | ○ | • |
| 29–32 | − | − | − | ○ | • | • | ○ |

$d_C$

C.P.

ABCD = a + −

A B C D

consider alternative expt.

A,B,C,D: control

a: noise

use $2^{5-1}_V$ with I=ABCDa

$\bullet : \mathbf{I} = ABCab$, $\circ : \mathbf{I} = -ABCab$,

v **Reading**: textbook, 11.6, 11.7

# Comparison of Cross Arrays and Single Arrays

special case

cross array : $I=ABC=abc=ABCabc$, $2^{6-2}_{III}$

eligible effect
A, B, C, a, b, c

clear
Aa, Ab, Ac, Ba, Bb, Bc, Ca, Cb, Cc.
A, B, C, a, b, c

minimum aberration

ab, c, Aa, Ba, Ca
AB, AC, ABC

A, B, C, Ab, Bb, Cb, Ac, Bc, Cc

- Example 1 (continued) An alternative is to choose a single array $2^{6-2}_{IV}$ design with $\mathbf{I}=\mathbf{ABCa}=\mathbf{ABbc}=\mathbf{abcC}$. This is not advisable because no 2fi's are clear and only main effects are clear. (Why? We need to have some clear control-by-noise 2fi interactions for robust optimization.) A better one is to use a $2^{6-2}_{III}$ design with $\mathbf{I}=\mathbf{ABCa}=\mathbf{abc}=\mathbf{ABCbc}$. It has 9 clear effects: $A,B,C,Ab,Ac,Bb,Bc,Cb,Cc$ (3 control main effects and 6 control-by-noise interactions).

☆ inadequency of resolution & minimum aberration (single array) ← they do not distinguish the two types of factors. ← Recall: another example treatment & block factors

☆ modification of effect ordering principle

$\boxed{C==N=C\times N} \gg \boxed{C\times C==C\times C\times N} \gg \boxed{N\times N}$

- **Reading**: textbook, 11.8