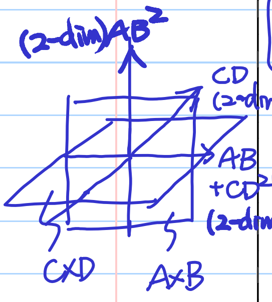
model: $y \sim A+B+C+D+(AB+CD^2)+AB^2+\cdots+\varepsilon$ --- (*)

assume 3fi or higher order interactions negligible

constant variance

ANOVA Table for Strength Location



Source	Degrees of Freedom	Sum of Squares	Mean Squares	F	p-value
A (4-dim A×B)	2	34621746	17310873	85.58	0.000
B	2	938539	469270	2.32	0.108
$AB = CD^2$	2	★2727451	1363725	6.74	0.002
AB^2 (4-dim C×D)	2	570795	285397	1.41	0.253
C	2	9549481	4774741	23.61	0.000
$AC = BD^2$	2	★2985591	1492796	7.38	0.001
AC^2 (A×C)	2	886587	443294	2.19	0.122
$BC = AD^2$	2	427214	213607	1.06	0.355
BC^2	2	21134	10567	0.05	0.949
D	2	4492927	2246464	11.11	0.000
AD	2	263016	131508	0.65	0.526
BD (B×D)	2	205537	102768	0.51	0.605
CD	2	245439	122720	0.61	0.549
residual	54	10922599	202270		

Note: cannot write the ANOVA into multi-way layout.

Analysis of Strength Location, Seat-Belt Experiment

- In equation (2), the 26 degrees of freedom in the experiment were grouped into 13 sets of effects. The corresponding ANOVA table gives the SS values for these 13 effects.

  alias set

  project y onto each 2-dim space (∵ orthogonality)

- Based on the p values in the ANOVA Table, clearly the factor A, C and D main effects are significant.
- Also two aliased sets of effects are significant, $AB = CD^2$ and $AC = BD^2$.
- These findings are consistent with those based on the main effects plot and interaction plots. In particular, the significance of AB and CD^2 is supported by the $A \times B$ and $C \times D$ plots and the significance of AC and BD^2 by the $A \times C$ and $B \times D$ plots.

same information appeared in different plots

$AB=CD^2$	not significant
AB^2	sig.
CD	not sig.

model: $y_{\underset{\sim}{x}} = \mu_{\underset{\sim}{x}} + \epsilon_{\underset{\sim}{x}}$, $Var(\epsilon_{\underset{\sim}{x}}) = \sigma_{\underset{\sim}{x}}^2$ ← not assumed constant.

Analysis of Strength Dispersion (i.e., $\ln s^2$) Data

$\mu_{\underset{\sim}{x}}$

$\bar{y}_{\underset{\sim}{x}} \sim \boxed{A+B+C+D+AB+AB^2+\cdots} + \epsilon'$ ← analysis for $\mu_{\underset{\sim}{x}}$ under the model (*) in LN p.1-26 should have similar conclusion

assume constant variance

The corresponding strength main effects plot and interaction plots are displayed in Figures 3 and 4.

$\ln(s_{\underset{\sim}{x}}^2) \sim \boxed{A+B+C+D+AB+AB^2+\cdots} + \epsilon''$

$\sigma_{\underset{\sim}{x}}^2$

may use $A_\ell, A_q, B_\ell, B_q, \ldots, (AB)_{\ell\ell}, \ldots$ (later lecture)

① $y_{\underset{\sim}{x}}$ data has replicates
② $\bar{y}_{\underset{\sim}{x}}, \ln(s_{\underset{\sim}{x}}^2)$ data have no replicates

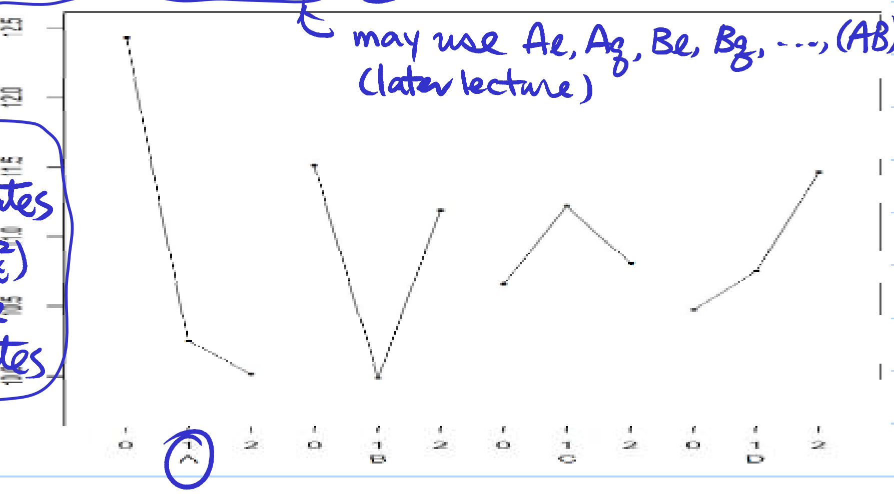


Figure 3: Main Effects Plot of Strength Dispersion, Seat-Belt Experiment

Interaction Plots of Strength Dispersion

Q: What effects in the ME plots & interaction plots can be declared as significant?

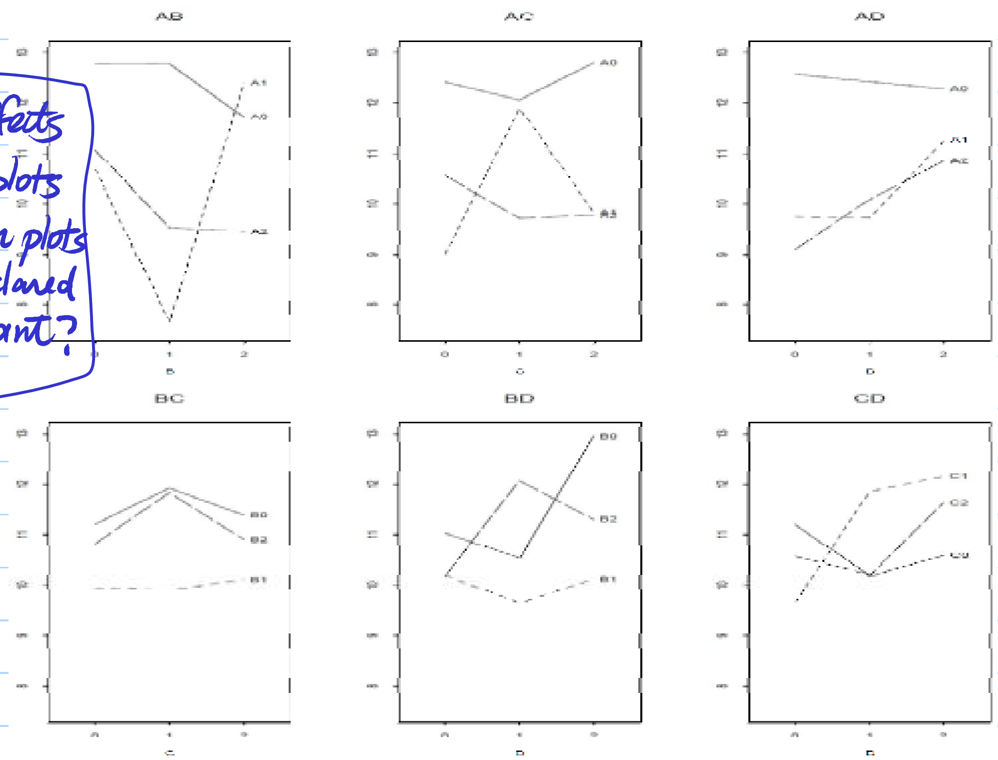


Figure 4: Interaction Plots of Strength Dispersion, Seat-Belt Experiment

Recall: $Z = X\beta + \boxed{\varepsilon}$ ← constant variance

$cov(\hat{\beta}) = (X^TX)^{-1}\sigma^2$

if $(X^TX)^{-1} \propto I \Rightarrow \hat{\beta}_i$'s are indep. Normal with constant variance.

Half-Normal Plots

- Since there is no replication for the dispersion analysis, ANOVA cannot be used to test effect significance.

- Instead, a half-normal plot can be drawn as follows. The 26 df's can be divided into 13 groups, each having two df's. These 13 groups correspond to the 13 rows in the ANOVA table of page 26. → mutually orthogonal & each 2-dim space.

- The two degrees of freedom in each group can be decomposed further into a linear effect and a quadratic effect with the contrast vectors $\frac{1}{\sqrt{2}}(-1, 0, 1)$ and $\frac{1}{\sqrt{6}}(1, -2, 1)$, respectively, where the values in the vectors are associated with the $\ln s^2$ values at the levels (0, 1, 2) for the group. Why? Helmert coding

- Because the linear and quadratic effects are standardized and orthogonal to each other, these 26 effect estimates can be plotted on the half-normal probability scale as in Figure 5.

$Z \sim \overbrace{A}^{A_l\ A_q} + B + C + D + \overbrace{AB}^{(AB)_l\ (AB)_q} + AB^2 + \cdots + \varepsilon$

	0	l	q	
A	0	-1	1	1
B	1	0	-2	1
AB	2	1	1	1
AB^2				
⋮				

orthogonal

Half-Normal Plot

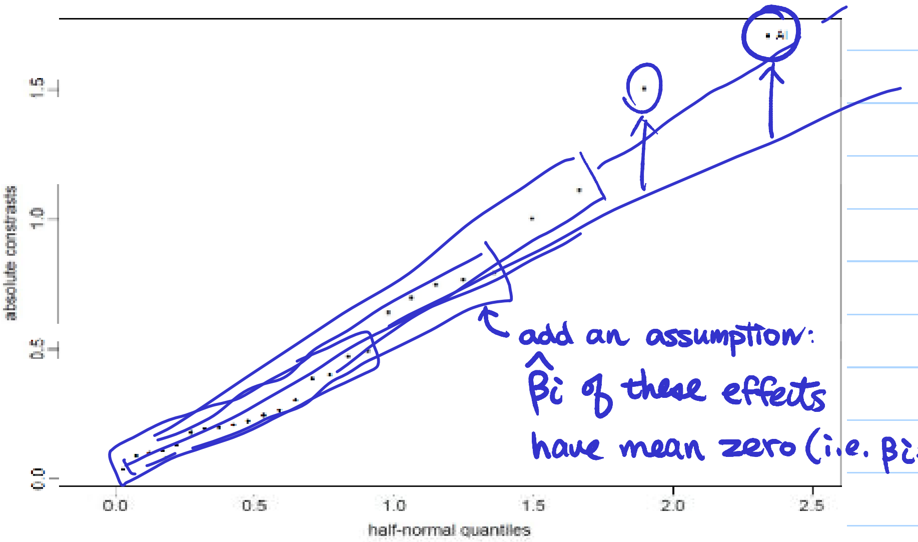


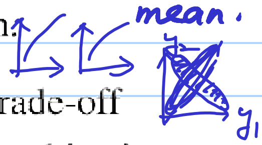
Figure 5: Half-Normal Plot of Strength Dispersion Effects, Seat-Belt Experiment

check whether $\hat{\beta}_i$'s in ~~these~~ this case still satisfy the model of Lenth's method

Informal analysis of the plot suggests that the factor A linear effect may be significant. This can be confirmed by using Lenth's method. The t_{PSE} value for the A linear effect is 3.99, which has a p value of 0.003(IER) and 0.050 (EER).

Analysis Summary

- A similar analysis can be performed to identify the flash location and dispersion effects. See Section 6.5 of WH book.
- We can determine the optimal factor settings that maximize the *strength location* by examining the main effects plot and interaction plots in Figures 1 and 2 that correspond to the significant effects identified in the ANOVA table.
  exercise
  → alternative: use final fitted model.
- We can similarly determine the optimal factor settings that minimize the *strength dispersion, the flash location* and *flash dispersion*, respectively.
- The most obvious findings: level 2 of factor A be chosen to maximize strength while level 0 of factor A be chosen to minimize flash.
  → most significant factor for strength & flash mean.



- There is an obvious conflict in meeting the two objectives. Trade-off strategies for handling multiple characteristics and conflicting objectives need to be considered (See Section 6.7 of WH).

v Reading: textbook, 6.5

An Alternative Analysis Method : Linear-Quadratic System

Recall: previous analyses, mainly for qualitative factors

In the seat-belt experiment, the factors A, B and C are $\boxed{\text{quantitative}}$. The two degrees of freedom in a quantitative factor, say A, can be decomposed into the linear and quadratic components.

Letting y_0, y_1 and y_2 represent the observations at level 0, 1 and 2, then the *linear effect* is defined as

μ_0 μ_1 μ_2

parametric definition of the effect → $\beta_{A\ell} \equiv \mu_2 - \mu_0$

$$\hat{\beta}_{A\ell} = y_2 - y_0$$

and the *quadratic effect* as

depend on the design

$\beta_{Aq} \equiv (\mu_2 + \mu_0) - 2\mu_1$

$$\hat{\beta}_{Aq} = (y_2 + y_0) - 2y_1,$$

which can be re-expressed as the difference between two consecutive linear effects $(y_2 - y_1) - (y_1 - y_0)$.

→ $(\mu_2 - \mu_1) - (\mu_1 - \mu_0)$

increasing/decreasing rate change.

Under full factorial 3^k

$\underset{\sim}{x} \rightarrow E(y_{\underset{\sim}{x}}) = \mu_{\underset{\sim}{x}}$

$\mu \equiv [\mu_{\underset{\sim}{x}}] = E[y_{\underset{\sim}{x}}] \equiv X_F \beta$

$X_F^T = X_F^{-1}$

$\Rightarrow \beta = X_F^T \mu$

μ_0 : average of $\mu_{\underset{\sim}{x}}$ with $A=0$

μ_1 : " " " " $A=1$

μ_2 : " " " " $A=2$

$Y = X\beta + \varepsilon$

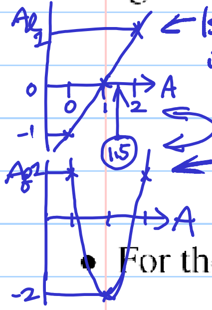
$\hat{\beta} = (X^T X)^{-1} X^T Y$

if $(X^T X) \propto I \Rightarrow \hat{\beta} \propto X^T Y$

Linear and Quadratic Effects

main.

Mathematically, the linear and quadratic effects are represented by two mutually orthogonal vectors:



1st order polynomial of A.

base function

2nd-order polynomial of A.

scaling

$$
\begin{aligned}
A_l &= \frac{1}{\sqrt{2}}(-1,0,1), \\
A_q &= \frac{1}{\sqrt{6}}(1,-2,1).
\end{aligned}
\quad (4)
$$

A		A_l	A_q
y_0	0	-1	1
y_1	1	0	-2
y_2	2	1	1

coding used in model matrix X

$y_2 - y_0$

$(y_2 + y_0) - 2y_1$

- For the sake of brevity, they are also referred to as the l and q effects.
- The scaling constants $\sqrt{2}$ and $\sqrt{6}$ yield vectors with unit length. e.g. in full factorial
- The linear (or quadratic) effect is obtained by taking the inner product between A_l (or A_q) and the vector $\mathbf{y} = (y_0, y_1, y_2)$. For factor B, B_l and B_q are similarly defined.

Linear and Quadratic Effects (contd)

interaction.

$(AB)_{ll}, (AB)_{lq}$
$(AB)_{ql}, (AB)_{qq}$

AB: $(AB)_l$, $(AB)_q$
AB^2: $(AB^2)_l$, $(AB^2)_q$

- Then the four degrees of freedom in the $\boxed{A \times B}$ interaction can be decomposed into four mutually orthogonal terms: $(AB)_{ll}, (AB)_{lq}, (AB)_{ql}, (AB)_{qq}$, which are defined as follows: for $i, j = 0, 1, 2$,

	A	B	A_l	A_q	B_l	B_q	$(AB)_{ll}$ ...	AB	$(AB)_l$	$(AB)_q$
μ_{00}	0	0	-1	1	-1	1	1	0	-1	1
μ_{01}	0	1	-1	1	0	-2	0	1	0	-2
	0	2	-1	1	1	1	-1	2	1	1
	1	0	0	-2	-1	1	0	1	0	-2
	1	1	0	-2	0	-2	0	2	1	1
	1	2	0	-2	1	1	0	0	-1	1
	2	0	1	1	-1	1	-1	2	1	1
	2	1	1	1	0	-2	0	0	-1	1
μ_{22} = $E(y_{22})$	2	2	1	1	1	1	1	1	0	-2

① c.p. ② ③ idea behind the coding: polynomial approximation

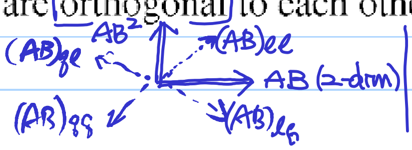
$$
\begin{aligned}
(AB)_{ll}(i,j) &= A_l(i)B_l(j), \quad \text{: order = 2} \\
(AB)_{lq}(i,j) &= A_l(i)B_q(j), \quad \text{: order = 3} \\
(AB)_{ql}(i,j) &= A_q(i)B_l(j), \quad \text{: order = 3} \\
(AB)_{qq}(i,j) &= A_q(i)B_q(j). \quad \text{: order = 4}
\end{aligned}
\quad (5)
$$

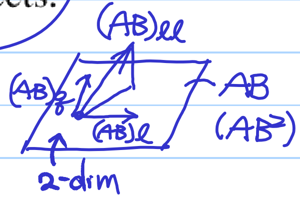
(model matrix) orthogonal in full factorial, but may not in fractional factorial

They are called the **linear-by-linear**, **linear-by-quadratic**, **quadratic-by-linear** and **quadratic-by-quadratic** interaction effects. They are also referred to as the $l \times l$, $l \times q$, $q \times l$ and $q \times q$ effects.

- It is easy to show that they are $\boxed{\text{orthogonal}}$ to each other.

$\{AB, AB^2\}$ & $\{(AB)_{ll}, (AB)_{lq}, (AB)_{ql}, (AB)_{qq}\}$ span the same space.





$\hat{\beta} = (X^TX)^{-1}X^TY$

$(X^TX) \propto I \Rightarrow \hat{\beta} \propto X^TY$

Linear and Quadratic Effects (contd)

Using the nine level combinations of factors A and B, $y_{00}, \ldots, y_{22}$ given in Table 5, the contrasts $(AB)_{ll}$, $(AB)_{lq}$, $(AB)_{ql}$, $(AB)_{qq}$ can be expressed as follows:

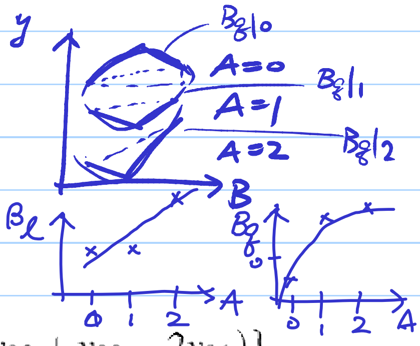
ll : parametric definition of effect.

$(AB)_{ll}$: $\frac{1}{2}\{\underbrace{(y_{22} - y_{20})}_{B_l|A=2} - \underbrace{(y_{02} - y_{00})}_{B_l|A=0}\}$,

$(AB)_{lq}$: $\frac{1}{2\sqrt{3}}\{\underbrace{(y_{22} + y_{20} - 2y_{21})}_{B_q|A=2} - \underbrace{(y_{02} + y_{00} - 2y_{01})}_{B_q|A=0}\}$,

$(AB)_{ql}$: $\frac{1}{2\sqrt{3}}\{(y_{22} + y_{02} - 2y_{12}) - (y_{20} + y_{00} - 2y_{10})\}$,

$(AB)_{qq}$: $\frac{1}{6}\{\underbrace{(y_{22} + y_{20} - 2y_{21})}_{B_q|A=2} - 2\underbrace{(y_{12} + y_{10} - 2y_{11})}_{B_q|A=1} + \underbrace{(y_{02} + y_{00} - 2y_{01})}_{B_q|A=0}\}$.



- An $(AB)_{ll}$ interaction effect measures the difference between the conditional linear B effects at levels 0 and 2 of factor A.
- A significant $(AB)_{ql}$ interaction effect means that there is curvature in the conditional linear B effect over the three levels of factor A.
- The other interaction effects $(AB)_{lq}$ and $(AB)_{qq}$ can be similarly interpreted.

• can be similarly extended to higher-order interactions.