(1, 2pts) This is a 2^3 design. In the design, all the effects are mutually *orthogonal* and the effect vectors in the model matrix have lengths of 8. So,

$$\begin{aligned} R^2 &= 1 - \frac{\hat{\sigma}^2 \times (8-4)}{(5^2\|x_1\|^2 + 8^2\|x_2\|^2 + (-3)^2\|x_3\|^2) + (\hat{\sigma}^2 \times (8-4))} \\ &= 1 - \frac{3 \times (8-4)}{(5^2 + 8^2 + (-3)^2) \times 8 + 3 \times (8-4)} = 1 - \frac{12}{784+12} = 98.49\%. \end{aligned}$$

Because of the high R^2, a first-order model seemed to be an adequate fit for the experimental region, based on the 8-run experiment.

(2, 1pt) $\mathbf{v}=(5,8,-3)/\sqrt{5^2+8^2+(-3)^2} = \left(\frac{5}{7\sqrt{2}}, \frac{8}{7\sqrt{2}}, \frac{-3}{7\sqrt{2}}\right)$.

(3, 1pt) The experimental region is $[-1,1]^3$. This is a steepest ascent problem so that λ is a positive value. When $\lambda = \frac{1}{8/(7\sqrt{2})} = 1.2374$, the point $\lambda \cdot \mathbf{v} = \left(\frac{5}{8}, 1, \frac{-3}{8}\right)$ is located on the boundary.

(4, 1pt) The coordinates of the point in the coded variables is $\left(\frac{5}{8}, 1, \frac{-3}{8}\right)$. It corresponds to $\left(105 + 5 \times \frac{5}{8},\ 1.5 + 0.5 \times 1,\ 62.5 - 12.5 \times \frac{3}{8}\right) = (108.125\ ^\circ C, 2\ \text{hour}, 57.8125\ \text{psi})$ in the natural variables.

(5, 1pt) Adding center runs to the 2^3 design can (i) help the estimation of error variance, and (ii) allow the study of the overall curvature effect.

(6, 2pts) For the 12-run design, the model matrix in the coded form is given below:

intercept	x_1	x_2	x_3	x_{12}	x_{13}	x_{23}	x_{123}	x_{oce}
1	−1	−1	−1	1	1	1	−1	1
1	−1	−1	1	1	−1	−1	1	1
1	−1	1	−1	−1	1	−1	1	1
1	−1	1	1	−1	−1	1	−1	1
1	1	−1	−1	−1	−1	1	1	1
1	1	−1	1	−1	1	−1	−1	1
1	1	1	−1	1	−1	−1	−1	1
1	1	1	1	1	1	1	1	1
1	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0

where x_{oce} represents the overall curvature effect. It can be easily checked that these factorial effects are mutually orthogonal.

(7, 3pts) The test statistics is

$$\frac{|\bar{y}_f - \bar{y}_c|}{\hat{\sigma}\sqrt{\frac{1}{n_f} + \frac{1}{n_c}}} = \frac{|100 - 125|}{\hat{\sigma}\sqrt{\frac{1}{8} + \frac{1}{4}}},$$

where $\hat{\sigma}$ can be estimated using $\sqrt{3.2} = 1.7888$ (with 3 degrees of freedom) or $\sqrt{\frac{3\times4+3.2\times3}{7}} = 1.7566$ (with 7 degrees of freedom). The corresponding test statistics give the value of 22.8218 ($> t_{3,0.975} = 3.182$) and 23.2406 ($> t_{7,0.975} = 2.365$), respectively. Either t-test rejects the null hypothesis of no curvature effect. We may conclude that a first-order model is inadequate for the experimental region.

(8, 2pts) Becase the 2^3 design and the center runs were performed at different times, we might want to add a block effect to measure the systematic difference between them (i.e., the 2^3 design in the first block and the center runs in the second block). The block effect is *confounded* with the overall curvature effect (i.e., $|\bar{y}_f - \bar{y}_c|$ estimates the joint effect of the blocks and the overall curvature). It is possible that the rejection of the no overall curvature effect in problem (7) is due to a significant block effect. In this case, the conclusion given in problem (7) becomes more questionable.

(9, 1pt) The levels of x1 and x2 are equally spaced. For x1, (5300, 6800, 8300) is coded as $(-1, 0, 1)$ respectively and for x2, (0.000, 0.006, 0.012) is coded as $(-1, 0, 1)$ respectively. So, the design matrix is:

Run	x1	x2
1	−1	−1
2	1	−1
3	−1	1
4	1	1
5	0	−1
6	0	1
7	−1	0
8	1	0
9	0	0
10	0	0
11	0	0

(10, 2pts) This is a central composite deisng (CCD) of two factors with $\alpha = 1$ and three center points.

(11, 2pts) The t-test for x22 is insignificant. We can remove it from the model. Because of orthogonality (a consequence of using the CCD given above), the removal of the effect x22 does not affect the estimates of β_{x1}, β_{x2}, and β_{x12}. For β_0 and β_{x11}, their estimates will be slightly changed. The new estimates of β_0 and β_{x11} can be obtained using the alias matrix and:

$$\begin{bmatrix} \hat{\beta}'_0 \\ \hat{\beta}'_{\text{x11}} \end{bmatrix} = \begin{bmatrix} \hat{\beta}_0 \\ \hat{\beta}_{\text{x11}} \end{bmatrix} + (\mathbf{X}_1^T\mathbf{X}_1)^{-1}\mathbf{X}_1^T\mathbf{X}_2\hat{\beta}_{\text{x22}},$$

where $\hat{\beta}_0$, $\hat{\beta}_{\text{x11}}$, and $\hat{\beta}_{\text{x22}}$ are the original coefficient estimates of intercept, x11, and x22, respectively, $\mathbf{X}_1$ is the 11×2 matrix containing the intercept and the x11 effect, and $\mathbf{X}_2$ is the 11×1 matrix containing only the x22 effect. The resulting fitted model is:

$$\hat{y} = 1.6780 + 0.6500 \times \text{x1} - 0.2883 \times \text{x2} - 0.3000 \times \text{x12} + 0.2287 \times \text{x11}.$$

(12, 2pts) The stationary point is

$$-\frac{1}{2}\mathbf{B}^{-1}\mathbf{b} = \begin{bmatrix} -0.9610 \\ 0.7015 \end{bmatrix},$$

where

$$\mathbf{B} = \begin{bmatrix} 0.2287 & -0.1500 \\ -0.1500 & 0 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 0.6500 \\ -0.2883 \end{bmatrix}.$$

The stationary point is located on the upper-left corner of the experimental region and is very close to the boundary.

(13, 3pts) The eigenvalues of $\mathbf{B}$ are 0.3030 and -0.0743. The corresponding standardized eigenvectors are respectively

$$e_1 = \begin{bmatrix} -0.8962 \\ 0.4437 \end{bmatrix} \text{ and } e_2 = \begin{bmatrix} -0.4437 \\ -0.8962 \end{bmatrix}.$$

So, the first canonical variable is

$$v_1 = (-0.8962) \times (\text{x1} - (-0.9610)) + 0.4437 \times (\text{x2} - 0.7015),$$

and the second canonical variable is

$$v_2 = (-0.4437) \times (\text{x1} - (-0.9610)) + (-0.8962) \times (\text{x2} - 0.7015).$$

The two eigenvalues have opposite signs, but the second eigenvalue is relatively smaller (in absolute value) than the first one. Because the stationary point is in the experimental region, this is more like a case of *stationary ridge system.* This conclusion is also supported by the contour plot given in the question sheet.

(14, 2pts) From the eigenvalues and eigenvectors of $\mathbf{B}$, and the contour plot, we know that the fitted response surface increases from the stationary point (on upper-left corner of the experimental region) in the direction of $-e_1$ (toward the lower-right conner). We can write the fitted response surface in the canonical form as follows:

$$\hat{y} = \hat{y}_s + (0.3030) \times v_1^2 + (-0.0743) \times v_2^2, \qquad \text{(I)}$$

where

$$\begin{aligned} \hat{y}_s &= 1.6780 + 0.6500 \times (-0.9610) - 0.2883 \times 0.7015 \\ &\quad -0.3000 \times (-0.9610) \times 0.7015 + 0.2287 \times (-0.9610)^2 \\ &= 1.2645. \end{aligned}$$

Set the $\hat{y}$ in (I) to 2.8, v_2 to zero and solve for v_1, we get the solution

$$2.8 = 1.2645 + (0.3030)v_1^2 \Rightarrow |v_1| = 2.2511.$$

The triangular region bounded by the line

$$-2.2511 = v_1 = (-0.8962) \times (\text{x1} - (-0.9610)) + 0.4437 \times (\text{x2} - 0.7015),$$

and the lines x1=1, x2=−1 gives $\hat{y} \geq 2.8$. More specifically, the triangular region is defined by connecting the three points (1, −0.4111), (−1, 0.7084), and (1, −1)

(15, 1.5pts) Yes. Note that the aliasing $A = B$ in the cube portion can be dealiased by adding the axial points.

(16, 1.5pts) No. The defining relation $I = AB$ in the cube portion causes the aliasing $AC = BC$. The aliasing cannot be removed by adding center points and/or axial points.

(17, 1.5pts) No, because $ABCD$ is a word of length 4.

(18, 1.5pts) No. When $\alpha = \sqrt{k}$, we need center point(s) to remove the linear dependence between the quadratic effects and the intercept.

(19, 2pts) The run size N must be divisible by 6 and 9. The smallest N is 18. However, the OA with 18 runs can accommodate at most one 2-level factor and seven 3-level factors. But, we have 11 3-level factors in the case. On the other hand, we can also find that there are 23 ($= 1 + 2 \times 11$) factorial main effects in the model and 18 runs are not enough to simultaneously estimate all the factorial main effects. Therefore, the smallest N must be at least 36.

(20, 1pt) The run size N must be divisible by 6, 8, 9, and 12. The smallest N is 72.

(21, 2pts) If we change the factors from $(2^1 3^4 4^1)$ to $(2^1 3^5)$, the smallest N is 18. If we change the factors from $(2^1 3^4 4^1)$ to $(2^2 3^3 4^1)$, the smallest N is still 72. The one with a smaller run size is the $OA(18, 2^1 3^5)$.

(22, 2pts) An OA with 16 runs can accommodate at most 15 2-level factors. By applying the method of replacement (3 2-level columns → one 4-level column) on the $OA(16, 2^{15})$, we can obtain an $OA(16, 2^9 4^2)$. By collapsing one 4-level factor in the $OA(16, 2^9 4^2)$ to a 3-level factor, we obtain an $OME(16, 2^9 3^1 4^1)$.

(23, 2pts) Let M_i be the class of submodels with $4 - i$ main effects and i two-factor interactions, where the interactions must have at least one parent factor among the $4 - i$ main effects. Then, $\#M_0 = \binom{19}{4} = 3876$, $\#M_1 = \binom{19}{3} \times \left(\binom{19}{2} - \binom{16}{2}\right) = 49419$, $\#M_2 = \binom{19}{2} \times \binom{(18+18-1)}{2} =$ 101745, $\#M_3 = \binom{19}{1} \times \binom{18}{3} = 15504$, and

$$\#M_0 + \#M_1 + \#M_2 + \#M_3 = 170544.$$

The percentage is

$$\frac{170544}{\binom{(19+\binom{19}{2})}{4}} = \frac{170544}{52602165} = 0.3242\%.$$