Exploitation of Nonlinearity

f: not a straight line (i.e., not a linear ME model)

Recall item 1 (LNp.10-7)

- Nonlinearity between y and $\mathbf{x}$ can be exploited for robustness if $\mathbf{x}_0$, nominal values of $\mathbf{x}$, are control-factor settings and deviations of $\mathbf{x}$ around $\mathbf{x}_0$ (i.e., $\mathbf{x}-\mathbf{x}_0$) are viewed as noise factors (called *internal noise*). Expand $\boxed{y=f(\mathbf{x})}$ around $\mathbf{x}_0$,

a variable $x_i = x_{i0}+\delta_i$
- a control factor $x_{i0} \equiv C_i$
- a noise factor $\delta_i = x_i - x_{i0} \equiv N_i$

(Recall. Errors in predictors, LM, LNp 9-7~8)

$\boxed{\underline{C}=\underline{x}_0}$

$$y \approx f(\mathbf{x}_0)+\sum_i \left(\frac{\partial f}{\partial x_i}\Big|_{x_{i0}}\right)\underbrace{(x_i - x_{i0})}_{=N_i}.$$

δ-method

- This leads to

This is a function of $\underline{C}$

$$\mathrm{Var}_N(y_{\underline{C}=\underline{x}_0}) = \sigma_{\underline{y}}^2 \approx \sum_i\left(\frac{\partial f}{\partial x_i}\Big|_{x_{i0}}\right)^2\sigma_i^2, \qquad (1)$$

variation of X_i around X_{i0} (i.e., $\mathrm{Var}(N_i)$)

where $\sigma_{\underline{y}}^2 = var(y)$, $\sigma_i^2 = var(x_i)$, each component x_i has mean x_{i0} and variance σ_i^2.

- From (1), it can be seen that $\sigma_{\underline{y}}^2$ can be reduced by choosing x_{i0} with a smaller slope $\frac{\partial f}{\partial x_i}\Big|_{x_{i0}}$. This is demonstrated in Figure 1. Moving the nominal value a to b can reduce $var(y)$ because the slope at b is more flat. This is a **parameter design** step.

- On the other hand, reducing the variation of x around a can also reduce $var(y)$. This is a **tolerance design** step.

item 4 in LNp.10-6

Exploitation of Nonlinearity to Reduce Variation

But, their means $E_N(y_{\underline{C}})$ also change: $f(a) \to f(b)$.

Hint. Use other control factors that have an effect on $E_N(y_{\underline{C}})$ but no effect on $\mathrm{Var}_N(y_{\underline{C}})$ to adjust the mean back to $f(a)$. (adjustment factors)

significant ME(N) & Int(C,N)

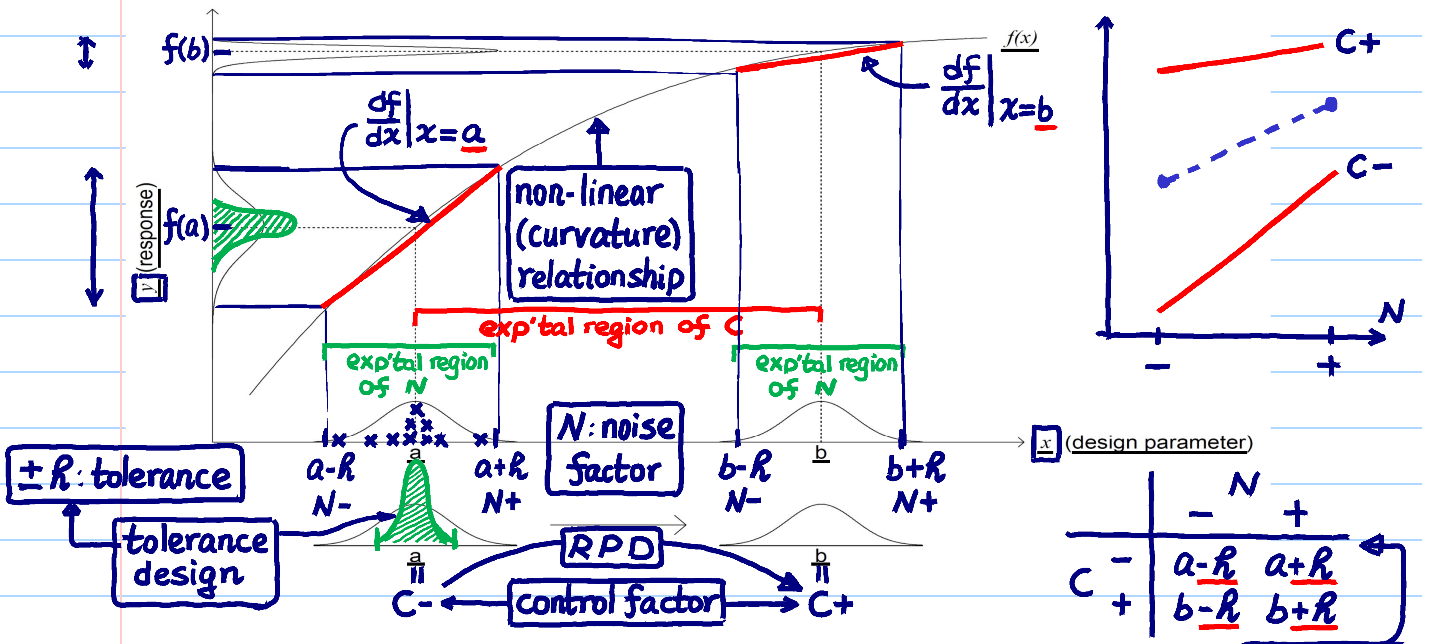


Figure 1: Exploiting the Nonlinearity of $f(x)$ to Reduce Variation

❖ **Reading**: textbook, 11.4

Cross Array and Location-Dispersion Modeling

the whole design matrix

$d_C \leftarrow n_C$ runs $\qquad d_N \leftarrow n_N$ runs

- Cross array = control array × noise array, ← $d_C \otimes d_N$ ← $n_C \times n_N$ runs
  - control array (or inner array) = array (design matrix) for control factors,
  - noise array (or outer array) = array (design matrix) for noise factors.

In the design matrix (DM), each control settings are "crossed" with same noise settings (e.g., DM in LN p.10-4~5)

cf. single array in LN p.10-29

- Location-dispersion modeling
  - use noise array
  - compute $\bar{y}_i$, s_i^2 based on the noise settings for the i^{th} control setting, ← same settings
  - use control array
  - analyze $\bar{y}_i$ (location), and $\ln s_i^2$ (dispersion), identify significant location and dispersion effects.

same analysis strategy used in CH4 & CH5 for data with replicates ← now, it's from d_N

Build 2 models:
$\bar{y} = X\beta_1 + \varepsilon$
$\ln s^2 = X\beta_2 + \varepsilon'$

model matrix X ← $AB\cdots$ d_C

$ab\cdots$ d_N: $y_{11}, \ldots, y_{1n_N}$ → $\bar{y}_1$, s_1^2

$\vdots$

$ab\cdots$ d_N: $y_{n_C 1}, \ldots, y_{n_C n_N}$ → $\bar{y}_{n_C}$, $s_{n_C}^2$

Two-step Procedures for RPD Optimization

← location & dispersion models

- **Two-Step Procedure for Nominal-the-Best Problem**

  (i) select the levels of the dispersion factors to minimize dispersion, ← use dispersion model for $\ln s_x^2$

  (ii) select the level of the adjustment factor (if exists) to bring the location on target. ← use location model for $\bar{y}_x$ (2)

- **Two-Step Procedure for Larger-the-Better and Smaller-the-Better Problems**

  (i) select the levels of the location factors to maximize (or minimize) the location, ← use location model for $\bar{y}_x$

  (ii) select the levels of the dispersion factors that are not location factors to minimize dispersion. ← use dispersion model for $\ln s_x^2$ (3)

- Note that the two steps in (3) are in reverse order from those in (2).
  Reason: It is usually harder to increase or decrease the response y in the latter problem, so this step should be the first to perform.

Analysis of Layer Growth Experiment

- From the $\bar{y}_i$ and $\ln s_i^2$ columns of Table 5 (LNp.10-14), compute the factorial effect estimates for location and dispersion respectively. (These numbers are not given in the textbook.) From the half-normal plots of these effects (Figure 2, LNp.10-15), D is significant for location and H, A for dispersion.

(exercise)

Note. There is no significant 2fi's in the 2 models. ∵ resolution = IV, if 2fi's are significant → dealiasing.

deposition time

$$\hat{y} = 14.352 + 0.402 x_D,$$
$$\ln \hat{s}^2 = -1.822 + 0.619 x_A - 0.982 x_H.$$

adjustment factor

- Two-step procedure:

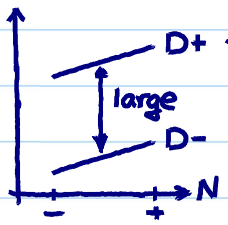
(i) Choose A at the "−" level (continuous rotation) and H at the "+" level (nozzle position = 6). → $(H+, A-)$

(ii) By solving

$$\hat{y} = 14.352 + 0.402 x_D = 14.5,$$

target value

choose $-1 < x_D = 0.368 < 1$. interpolation



Layer Growth Experiment: Analysis Results

Table 5: Means, Log Variances and SN Ratios, Layer Growth Experiment

A	B	C	D	E	F	G	H	$\bar{y}_i$	$\ln s_i^2$	$\ln \bar{y}_i^2$	$\hat{\eta}_i$
−	−	−	+	−	−	−	−	14.79	−1.018	5.389	6.41
−	−	−	+	+	+	+	+	14.86	−3.879	5.397	9.28
−	−	+	−	−	−	+	+	14.00	−4.205	5.278	9.48
−	−	+	−	+	+	−	−	13.91	−1.623	5.265	6.89
−	+	−	−	−	+	−	+	14.15	−5.306	5.299	10.60
−	+	−	−	+	−	+	−	13.80	−1.236	5.250	6.49
−	+	+	+	−	+	+	−	14.73	−0.760	5.380	6.14
−	+	+	+	+	−	−	+	14.89	−1.503	5.401	6.90
+	−	−	−	−	+	+	−	13.93	−0.383	5.268	5.65
+	−	−	−	+	−	−	+	14.09	−2.180	5.291	7.47
+	−	+	+	−	+	−	+	14.79	−1.238	5.388	6.63
+	−	+	+	+	−	+	−	14.33	−0.868	5.324	6.19
+	+	−	+	−	−	+	+	14.77	−1.483	5.386	6.87
+	+	−	+	+	+	−	−	14.88	−0.418	5.400	5.82
+	+	+	−	−	−	−	−	13.76	−0.418	5.243	5.66
+	+	+	−	+	+	+	+	13.97	−2.636	5.274	7.91

Control Factor; unreplicated; −0 ~ −5; 0.2 ~ 0.4

SN ratio $\hat{\eta}_i = \log\left(\frac{\bar{y}_i^2}{s_i^2}\right)$

Table 2 (LNp.10-4) Use 8 observations to calculate them

dc: 2^{8-4}_{IV} ← There are 15 df. available for studying effects

$I = -ABCD$
$= ABEF = -CDEF$
$= ACEG = -BDEG$
$= BCFG = -ADFG$
$= BCEH = -ADEH$
$= ACFH = -BDFH$
$= ABGH = -CDGH$
$= EFGH = -ABCDEFGH$
(check LNp10-4)

conceptual model:

$\bar{y}$ or $\ln s^2$

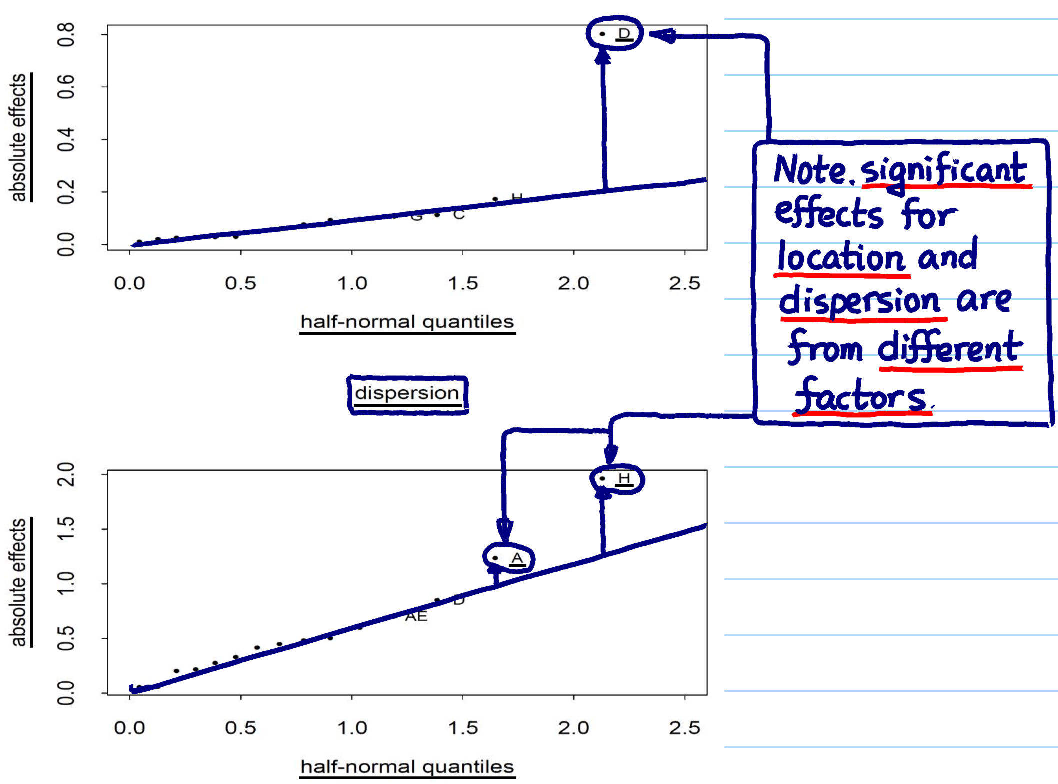
$$z = \beta_0 + \sum_i \beta_i (\text{factorial effect})_i + \varepsilon \quad \text{(no replicates)}$$

a joint effect

an alias set contributes an effect

⇒ estimate 15 effects → no df. left for residuals

Layer Growth Experiment: Plots



Alternative: Lenth's method

Figure 2: Half-Normal Plots of Location and Dispersion Effects, Layer Growth Experiment

Analysis of Leaf Spring Experiment

- From the $\bar{y}_i$ and $\ln s_i^2$ columns of Table 6 (LNp.10-17), compute the factorial effect estimates for location and dispersion respectively. Based on the half-normal plots (exercise) in Figure 3 (LNp.10-18), B, C and E are significant for location, C is significant for dispersion:

adjustment factor

$$\hat{y} = 7.6360 + 0.1106 x_B + 0.0881 x_C + 0.0519 x_E,$$

$$\ln \hat{s}^2 = -3.6886 + 1.0901 x_C.$$

alternative: use MSE = Var + bias² (check LNp 5-29) to choose a setting ⇒ compromising var & bias.

- Two-step procedure:

  (i) Choose C at $-$.

  (ii) With $x_C = -1$, $\hat{y} = 7.5479 + 0.1106 x_B + 0.0519 x_E$.

  can do confirm exp't

  * To achieve $\hat{y} = 8.0$, x_B and x_E must be chosen beyond $+1$ (e.g., $x_B = x_E = 2.78$). This is too drastic, and not validated by current data.

  extrapolation

  * An alternative is to select $x_B = x_E = x_C = +1$ (not to follow the two-step procedure), then $\hat{y}$=7.89 is closer to 8. (Note that $\hat{y} = 7.71$ with $B_+C_-E_+$.)

  * Reason for the breakdown of the 2-step procedure: its second step cannot achieve the target 8.0.

  exchange 1st & 2nd steps

Leaf Spring Experiment: Analysis Results

Table 6: Means and Log Variances, Leaf Spring Experiment

dc: 2^{4-1}_{IV} (8 runs)

I = BCDE

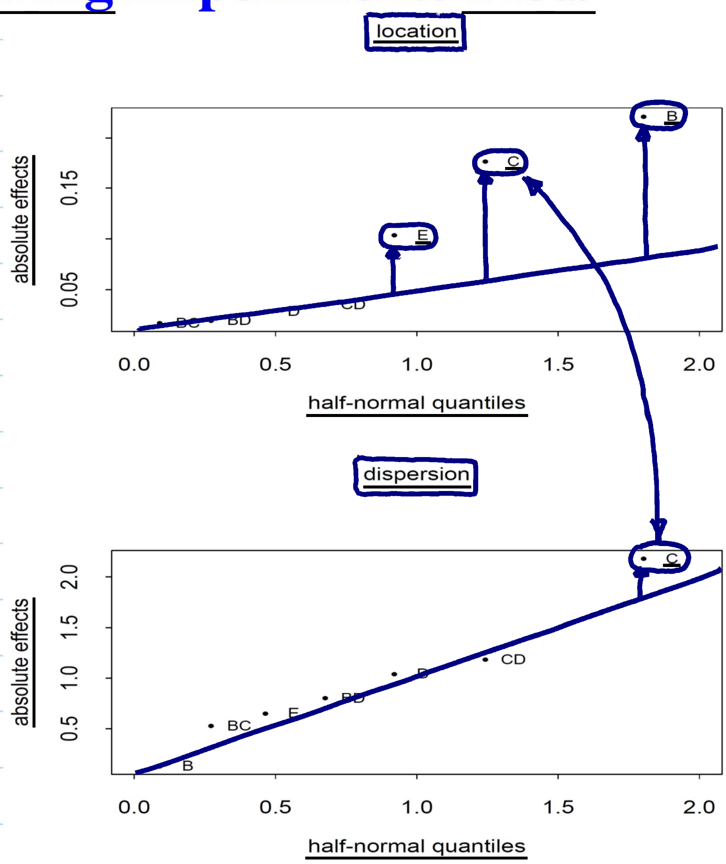
alias sets:

B = CDE
C = BDE
D = BCE
E = BCD
BC = DE
BD = CE
BE = CD

Control Factor				unreplicated	unreplicated
B	C	D	E	$\bar{y}_i$	$\ln s_i^2$
−	+	+	−	7.540	−2.4075
+	+	+	+	7.902	−2.6488
−	−	+	+	7.520	−6.9486
+	−	+	−	7.640	−4.8384
−	+	−	+	7.670	−2.3987
+	+	−	−	7.785	−2.9392
−	−	−	−	7.372	−3.2697
+	−	−	+	7.660	−4.0582

conceptual model: similar to what given in LNp.10-14

Leaf Spring Experiment: Plots



Alternative: Lenth's method

Figure 3: Half-Normal Plots of Location and Dispersion Effects, Leaf Spring Experiment

Response Modeling and Control-by-Noise Interaction Plots

cf → Location dispersion modeling

graphs in LNp.10-8

cf. ① may mask some important relationship btw $\underset{\sim}{C}$ & $\underset{\sim}{N}$ factors ② model ↔ approximation Q: Is it a good way to use $\bar{y}$, $\ln S^2$ as the response to build linear models?

Both control & noise factors are treatment factors

- Response Model: model y_{ij} directly in terms of control and noise main effects and control-by-noise interactions.
  - half normal plot of various effects.
  - regression model fitting, obtaining $\hat{y}$.

① $y \sim f(\underset{\sim}{C}, \underset{\sim}{N}) + \varepsilon$ ← constant variance; $f(\underset{\sim}{C}, \underset{\sim}{N}) = E(y|\underset{\sim}{C}, \underset{\sim}{N})$

② treat $\underset{\sim}{N}$ as r.v.

location → $E_{\underset{\sim}{N}}(f(\underset{\sim}{C}, \underset{\sim}{N}))$

dispersion → $Var_{\underset{\sim}{N}}(f(\underset{\sim}{C}, \underset{\sim}{N}))$

both are functions of $\underset{\sim}{C}$ only

Recall role of noise factors (LNp.10-1)
- its value is fixed in experiment
- its value is "random" in normal usage

graphical method (intuitive perception)

- Make control-by-noise interaction plots for significant effects in $\hat{y}$, choose **robust** control settings at which y has a flatter relationship with noise factors.

numerical method (easier for prediction)

- Compute $Var_N(\hat{y}_{\mathbf{x}})$ with respect to variation in the noise factors. Call $Var_N(\hat{y}_{\mathbf{x}})$ the **transmitted variance model**. Use it to identify control factor settings with small transmitted variance.

response variable for objective / modeling — no need to be identical

Q: What variable is more suitable to be the response in the fitted model? Apply approximation viewpoint.

Half-normal Plot, Layer Growth Experiment

- Define (L: Top-bottom, M: facets; LNp.10-4, 10-14)

$$\hat{\beta} \propto \begin{cases} M_l = (M_1 + M_2) - (M_3 + M_4), \\ M_q = (M_1 + M_4) - (M_2 + M_3), \\ M_c = (M_1 + M_3) - (M_2 + M_4), \end{cases}$$

coding

M	M_l	M_q	$M_c = M_l M_q$
1	−	+	−
2	−	−	+
3	+	−	−
4	+	+	+

3 main effects

benefit of this coding: orthogonality

similar to the coding for pseudo block factors in block FFD

- From Figure 4 (LNp.10-21), select the effects D, L, HL as the most significant effects.

  L causes larger variation

- How to deal with the next cluster of effects in Figure 4? Use **step-down multiple comparisons**.

The whole design matrix of $\underset{\sim}{C}$ & $\underset{\sim}{N}$ factors ($d_C \otimes d_N$) can be regarded as a $2^{(8+3)-4}_{IV}$ (128 runs) (3: L, M_l, M_q) with the defining contrast subgroup given in LNp.10-14.

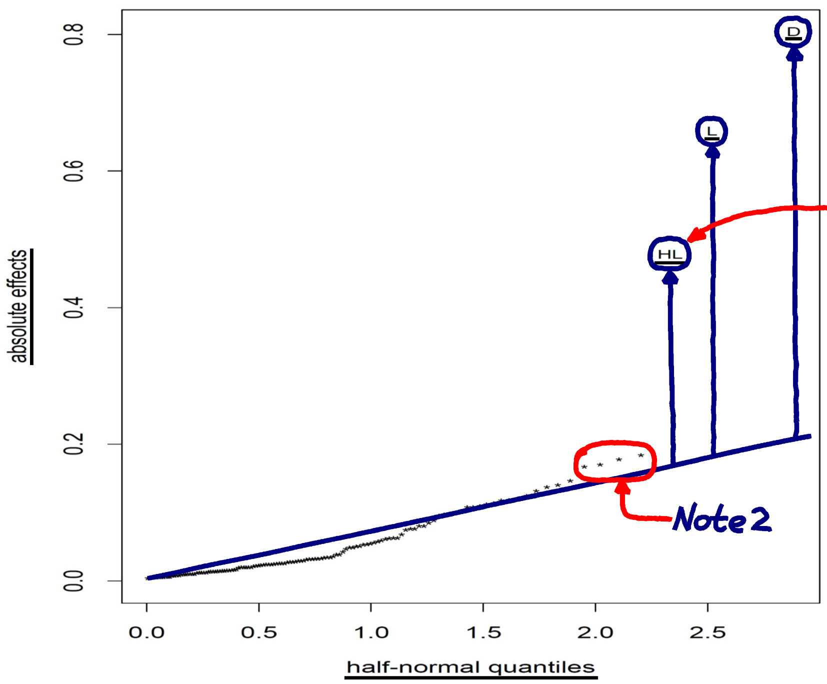
- After removing the top three points in Figure 4, make a half-normal plot (Figure 5, LNp.10-22) on the remaining points. The cluster of next four effects (M_l, H, CM_l, AHM_q) appear to be significant.

cf. → variation caused by L

conceptual model: similar to what given in LNp.10-4 LNp.10-14 (no replicate)

→ estimate factorial effects (exercise)

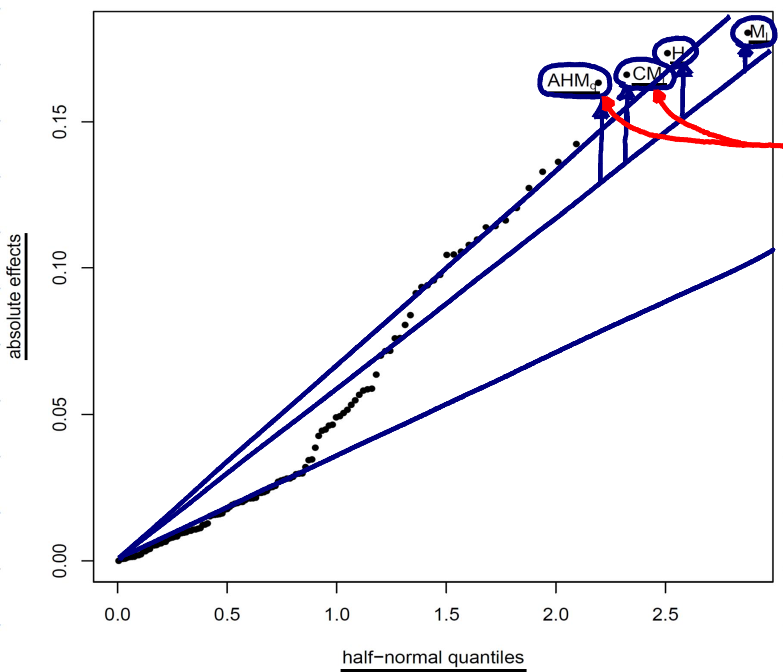
Half-normal Plot of Factorial Effects



Note 1. significant C×N interaction only involve the noise factor L

Figure 4: Half-Normal Plot of Response Model Effects, Layer Growth Experiment

Second Half-normal Plot of Factorial Effects



Note: significant C×N interaction only involve the noise factor M

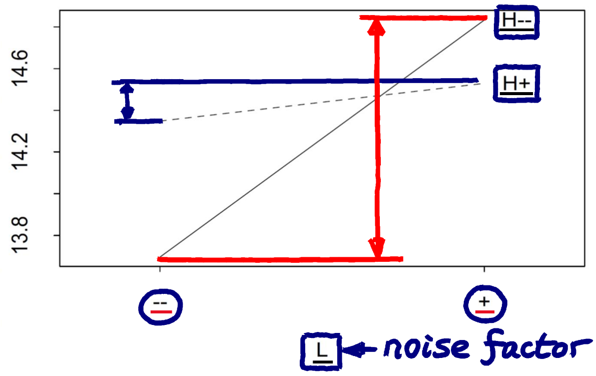
Figure 5: Second Half-Normal Plot of Response Model Effects, Layer Growth Experiment

Control-by-noise Interaction Plots

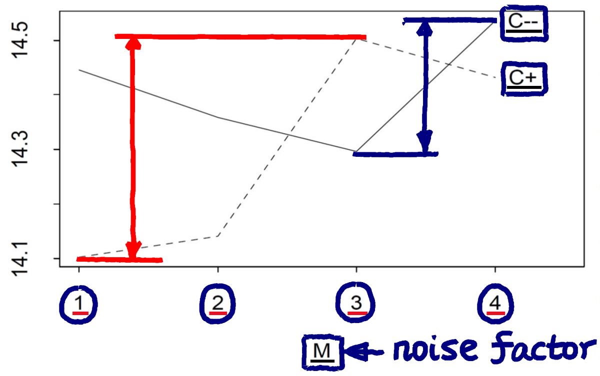
significant

HL, CM_ℓ, AHM_q

can reduce larger variation ⇒



⇒ H = +

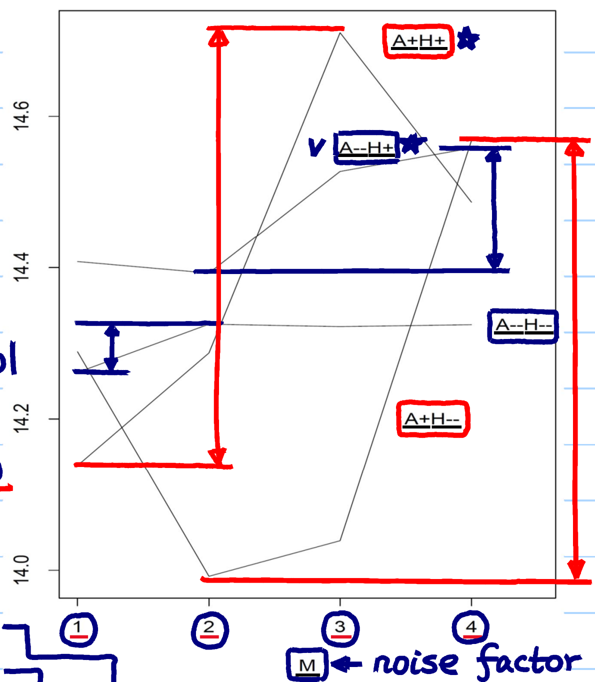


⇒ C = –

Figure 6: $H \times L$ and $C \times M$ Interaction Plots, Layer Growth Experiment

Three-factor Interaction Plot: $A \times H \times M$

✪ H×L and A×H×M interaction plots offer more physical interpretation about how the variation caused by the noise factors L & M is related to the control factors A & H. (cf. location-dispersion modeling)



from C×M interaction plot

from H×L interaction plot

Recommendation: (H+, C–, A–) $\xleftrightarrow{\text{cf.}}$ recommendation from dispersion model in LNp.10-13

Figure 7: $A \times H \times M$ Interaction Plot, Layer Growth Experiment

Response Modeling, Layer Growth Experiment

from half-normal plots in LNp.10-21~22

- The following model is obtained: can adjust $E_N(y|c)$

$$\hat{y} = 14.352 + 0.402 x_D + 0.087 x_H + 0.330 x_L - 0.090 x_{M_l}$$
$$-0.239 x_H x_L - 0.083 x_C x_{M_l} - 0.082 x_A x_H x_{M_q}. \qquad (4)$$

$E(y|c,N)$

What if treated as random? if treat x_L, x_{M_l}, x_{M_q} as random variables with mean 0

- Recommendations:

H:	$-$ (position 2)	to	$+$	(position 6)
A:	$+$ (oscillating)	to	$-$	(continuous)
C:	$+$ (1210)	to	$-$	(1220)

resulting in 37% reduction of thickness standard variation.

Transmitted Variance Model

P	L	x_L
1/2 → 1		$-$
1/2 → 2		$+$

P	M	x_{M_l}	x_{M_q}	$x_{M_l}x_{M_q} = x_{M_c}$
1/4 → 1		$-$	$+$	$-$
1/4 → 2		$-$	$-$	$+$
1/4 → 3		$+$	$-$	$-$
1/4 → 4		$+$	$+$	$+$

- Assume L, M_l and M_q are random variables, taking -1 and $+1$ with equal probabilities. This leads to

$$x_L^2 = x_{M_l}^2 = x_{M_q}^2 = x_A^2 = x_C^2 = x_H^2 = 1,$$
$$E(x_L) = E(x_{M_l}) = E(x_{M_q}) = 0,$$
$$Cov(x_L, x_{M_l}) = Cov(x_L, x_{M_q}) = Cov(x_{M_l}, x_{M_q}) = 0, \qquad (5)$$
$$Var(x_L) = Var(x_{M_l}) = Var(x_{M_q}) = 1.$$

x_L, x_{M_l}, x_{M_q}: indep. r.v.'s

- From (4) and (5), we have

$Var_N[E(y|c,N)]$

$$Var_N(\hat{y}_{\mathbf{x}}) = (.330 - .239 x_H)^2 Var(x_L) + (-.090 - .083 x_C)^2 Var(x_{M_l})$$
$$+ (.082 x_A x_H)^2 Var(x_{M_q})$$
$$= \text{constant} + (.330 - .239 x_H)^2 + (-.090 - .083 x_C)^2 + (0.082)^2 x_A^2 x_H^2$$
$$= \text{constant} - 2(.330)(.239) x_H + 2(.090)(.083) x_C$$
$$= \text{constant} - .158 x_H + .015 x_C.$$

dispersion model

location model

$E_N[E(y|c,N)] = 14.352 + 0.402 x_D + 0.087 x_H$

Assume only $x \in \{-1, +1\}$ is allowed.

Note. If ME M_q is in the model or x_A can be set $= 0$, this term can be used to reduce variance.

- Choose $H+$ and $C-$. But factor A is not present here. (Why? See explanation on textbook, p.532).

❖ **Reading**: textbook, 11.5

What if we add $\hat{\beta}_{M_q} \cdot x_{M_q}$ into model (4) in LNp.10-25? cf. optimal settings discussed in LNp.6-19