Techniques for Resolving Ambiguities in Aliased Effects

- Among the three factorial effects that feature in model (4) (LNp.6-17), B is clear and DQ is strongly clear.
- However, the term $x_B x_C x_Q$ is aliased with $x_D x_E x_Q$ (See bottom of LNp. 6-4). The following three techniques can be used to resolve the ambiguities.

For example, $I=BCDE$, $Z=\ln s^2$

significant:
$B=CDE$
$DQ=BCEQ$
$BCQ=DEQ$ (?)

insignificant:
$BC=DE$ e.g.
$BD=CE$
⋮

not reject $H_0: \beta_{BC}+\beta_{DE}=0$
⇒ usually conclude $\begin{cases}\beta_{BC}\approx 0\\ \beta_{DE}\approx 0\end{cases}$
Q: What if $\beta_{BC}\approx-\beta_{DE}$?

∵ not enough information

Use information form
演繹法

- Subject matter knowledge may suggest some effects in the alias set are not likely to be significant (or does not have a good physical interpretation).

Use information form assumption

- Or use effect hierarchy principle to assume away some higher order effects.

collect more information using expt
歸納法

- Or use a **follow-up experiment** to **de-alias** these effects. Two methods are given in section 5.4 of textbook.

Fold-over Technique

- Suppose the original experiment is based on a 2^{7-4}_{III} design with generators

# of factors
$\frac{1}{2^4}=\frac{1}{16}$ fraction
an alias set contains $2^4=16$ factorial effects
resolution
$2^3=8$ runs

$$d_1: \quad 4=12, \quad 5=13, \quad 6=23, \quad 7=123.$$

defining relations (4 generators)

defining contrast subgroup
$I=124=135=236=1237$
$=2345=1346=\cdots$
$=456=\cdots=1234567$

None of its main effects are clear.

- To de-alias them, we can choose another 8 runs (no. 9-16 in Table 4, LNp.6-23) with **reversed** signs for each of the 7 factors. This follow-up design d_2 has the generators

$$d_2: 4'=-1'2', 5'=-1'3', 6'=-2'3', 7'=1'2'3'.$$

$1'=-1, 2'=-2, \cdots, 7'=-7$
$-4'=(-1')(-2') \Rightarrow 4'=-1'2'$
$-5'=(-1')(-3') \Rightarrow 5'=-1'3'$
$-6'=(-2')(-3') \Rightarrow 6'=-2'3'$
$-7'=(-1')(-2')(-3')$
$\Rightarrow 7'=+1'2'3'$

With the extra degrees of freedom, we can introduce a new factor **8** (or a blocking variable) for run number 1-8, and **−8** for run number 9-16. See Table 4.

$I=8$
$-I=8'$
Why?

- The combined design $d_1 \pm d_2$ is a 2^{8-4}_{IV} design and thus all main effects are clear. (Its defining contrast subgroup is on textbook, p.227).

Augmented Design Matrix Using Fold-over Technique

Table 4: Augmented Design Matrix Using Fold-Over Technique

the combined design matrix $d_1 + d_2$

$2^{7-3}_{IV} = 2^4$ runs

I = 1237
= 2345
= 1346
= 1256
= 1457
= 2467
= 3567

resolution = IV
5 = 234
6 = 134
7 = 123
8 = 124

				d_1				I
Run	1	2	3	4=12	5=13	6=23	7=123	8
	1st design matrix							
1	−	−	−	+	+	+	−	+
2	−	−	+	+	−	−	+	+
3	−	+	−	−	+	−	+	+
4	−	+	+	−	−	+	−	+
5	+	−	−	−	−	+	+	+
6	+	−	+	−	+	−	−	+
7	+	+	−	+	−	−	−	+
8	+	+	+	+	+	+	+	+
Run	1′ = −1	2′ = −2	3′ = −3	4′ = −4 d_2	5′ = −5	6′ = −6	7′ = −7	8′ = −8
	2nd design matrix							
9	+	+	+	−	−	−	+	−
10	+	+	−	−	+	+	−	−
11	+	−	+	+	−	+	−	−
12	+	−	−	+	+	−	+	−
13	−	+	+	+	+	−	−	−
14	−	+	−	+	−	+	+	−
15	−	−	+	−	+	+	+	−
16	−	−	−	−	−	−	−	−

$2^{7-4}_{III} = 2^3$ runs

- 1st design matrix d_1

I = ~~124~~ = ~~135~~ = ~~236~~ = 1237
= 2345 = 1346 = ~~347~~
= 1256 = ~~257~~ = ~~167~~
= ~~456~~ = 1457 = 2467
= 3567 = ~~1234567~~
= ~~8~~ = 1248 = 1358
= 2368 = ~~12378~~ = ⋯
= 12345678

- 2nd design matrix d_2

I = −1′2′4′ = −1′3′5′ = −2′3′6′
= 1′2′3′7′ = 2′3′4′5′ = 1′3′4′6′
= ⋯ = −1′2′3′4′5′6′7′
= −8′ = 1′2′4′8′ = 1′3′5′8′
= 2′3′6′8′ = −1′2′3′7′8′ = ⋯
= 1′2′3′4′5′6′7′8′

Fold-over Technique: Version Two

- Suppose one factor, say **5**, is very important. We want to de-alias **5** and all 2fi's involving **5**.
- Choose, instead, the following 2^{7-4}_{III} design

$$d_3 : 4' = 1'2', 5' = -1'3', 6' = 2'3', 7' = 1'2'3'.$$

1′ = 1, 2′ = 2, 3′ = 3, 4′ = 4
5′ = −5, 6′ = 6, 7′ = 7

⇒ 2nd design matrix d_3

$$\begin{bmatrix} 1' & 2' & 3' & 4' & 5' & 6' & 7' \\ \| & \| & \| & \| & \| & \| & \| \\ 1 & 2 & 3 & 4 & -5 & 6 & 7 \\ | & | & | & | & | & | & | \end{bmatrix}$$

Then the combined design $d_1 + d_3$ is a 2^{7-3}_{III} design with the generators

$$d' : 4 = 12, 6 = 23, 7 = 123. \quad (5)$$

{1,2,3,5} forms a full factorial

a design favors factor 5

I = 1′2′4′ = −1′3′5′ = 2′3′6′
= 1′2′3′7′ = −2′3′4′5′
= 1′3′4′6′ = ⋯
= −1′2′3′4′5′6′7′

Since **5** does not appear in (5), **5** is strongly clear and all 2fi's involving **5** are clear. However, other main effects are not clear (see Table 5.7 in textbook, p.228, for $d_1 + d_3$).

⇒ the whole design matrix $d_1 + d_3$

$$\underbrace{\begin{aligned} I &= 124 = 236 = 1237 \\ &= 1346 = 347 = 167 \\ &= 2467 \end{aligned}}_{\text{“5” is not in any words.}}$$

Q: Why is it good?

- Choice between d_2 and d_3 depends on the priority given to the effects. → Note, effect hierarchy treats all factors as equally important.

Critique of Fold-over Technique

e.g I=ABC
A=BC
~~B=AC~~
~~C=AB~~

- Fold-over technique is not an efficient technique.
  - It requires doubling of the run size and can only de-alias a *specific* set of effects.
  - In practice, after analyzing the first experiment, a set of effects will emerge and need to be de-aliased.
  - It will usually require much *fewer* runs to de-alias a few effects.

e.g.,
A=BC ← I=ABC
DE=FG ← I=DEFG

significant alias sets

fold-over technique is not flexible enough to dealias any significant alias sets.

- A more efficient technique that does not have these deficiencies is the optimum design approach given in Section 5.4.2.

model matrix X must be known

★ define a real-valued criterion (e.g., D-, E-, A-, ⋯) on the Fisher information matrix of $\hat{\beta}$ (i.e., $[\text{cov}(\hat{\beta})]^{-1} = [\sigma^2 (X^TX)^{-1}]^{-1} \propto X^TX$) & optimize the criterion over all possible designs.

use smaller run size in the 2nd design matrix and totally focus on dealiasing significant alias sets

❖ **Reading**: textbook, 5.4.1

Use of Design Tables

- Minimum aberration (MA) designs are given in the tables in textbook, Appendix 5A.

LNp.6-12

If two designs are given for same k and p,

k: # of (treatment) factors; p: fraction

same if only one design is given.

⊖ the first is an MA design and

⊖ the second is better in having a larger number of clear effects.

Two tables are given on next two slides.

Note. ① MA criterion favors no particular factor (all factors are equally important)
② design not MA but having more clear effects usually favors some particular factors over the other factors.

- In Table 7 (LNp.6-28),

cf.

⊖ the first 2^{9-4} design has MA and 8 clear 2fi's, and

⊖ the second 2^{9-4} design is
  - ∗ the second best according to the MA criterion,
  - ∗ but has 15 clear 2fi's.

Using Rule (iii) in (2) on LNp.6-10, the second design is better because both have resolution IV (Details given on p. 234 of textbook).

- It is not uncommon to find a design with slightly worse aberration but more clear effects. Thus **the number of clear effects** should be used as a *supplementary criterion* to the MA criterion.

Table 6: 16-Run 2^{k-p} FFD $(k-p=4)$

← factors {1,2,3,4} form a 2^4 full factorial in the design matrix.

run size 2^4

(*k* is the number of factors and F&R is the fraction and resolution.)

P

k	F&R	Design Generators ← check LNp.6-8	Clear Effects
5	2_V^{5-1}	$5=1234$	all five main effects, all 10 2fi's
6	2_{IV}^{6-2}	$5=123$, $6=124$	all six main effects
6*	2_{III}^{6-2}	$5=12$, $6=134$ 9 clear effects →	3, 4, 6, 23, 24, 26, 35, 45, 56
7	2_{IV}^{7-3}	$5=123$, $6=124$, $7=134$	all seven main effects
8	2_{IV}^{8-4}	$5=123$, $6=124$, $7=134$, $8=234$	all eight main effects
9	Why? → 2_{III}^{9-5} ←	$5=123$, $6=124$, $7=134$, $8=234$, $9=1234$	none
10	2_{III}^{10-6}	$5=123$, $6=124$, $7=134$, $8=234$, $9=1234$, $t_0=34$	none
11	2_{III}^{11-7}	$5=123$, $6=124$, $7=134$, $8=234$, $9=1234$, $t_0=34$, $t_1=24$	none
12	2_{III}^{12-8}	$5=123$, $6=124$, $7=134$, $8=234$, $9=1234$, $t_0=34$, $t_1=24, t_2=14$	none
13	2_{III}^{13-9}	$5=123$, $6=124$, $7=134$, $8=234$, $9=1234$, $t_0=34$, $t_1=24, t_2=14$, $t_3=23$	none
14	2_{III}^{14-10}	$5=123$, $6=124$, $7=134$, $8=234$, $9=1234$, $t_0=34$, $t_1=24, t_2=14$, $t_3=23$, $t_4=13$	none
15	2_{III}^{15-11}	$5=123$, $6=124$, $7=134$, $8=234$, $9=1234$, $t_0=34$, $t_1=24, t_2=14$, $t_3=23$, $t_4=13$, $t_5=12$	none

---

Table 7: 32 Run 2^{k-p} FFD $(k-p=5, 6 \leq k \leq 11)$

factors {1,2,3,4,5} form a 2^5 full factorial in the design matrix.

(*k* is the number of factors and F&R is the fraction and resolution.)

k	F&R	Design Generators	Clear Effects
6	2_{VI}^{6-1}	$6=12345$	all six main effects, all 15 2fi's
7	2_{IV}^{7-2}	$6=123$, $7=1245$	all seven main effects, 14, 15, 17, 24, 25, 27, 34, 35, 37, 45, 46, 47, 56, 57, 67
8	2_{IV}^{8-3}	$6=123$, $7=124$, $8=1345$	all eight main effects, 15, 18, 25, 28, 35, 38, 45, 48, 56, 57, 58, 68, 78
9	2_{IV}^{9-4}	$6=123$, $7=124$, $8=125$, $9=1345$	all nine main effects, 19, 29, 39, 49, 59, 69, 79, 89
9	2_{IV}^{9-4}	$6=123$, $7=124$, $8=134$, $9=2345$	all nine main effects, 15, 19, 25, 29, 35, 39, 45, 49, 56, 57, 58, 59, 69, 79, 89
10	2_{IV}^{10-5}	10th factor $6=123$, $7=124$, $8=125$, $9=1345$, $t_0=2345$	all 10 main effects
10	2_{III}^{10-5}	$6=12$, $7=134$, $8=135$, $9=145$, $t_0=345$	3, 4, 5, 7, 8, 9, t_0, 23, 24, 25, 27, 28, 29, $2t_0$, 36, 46, 56, 67, 68, 69, $6t_0$
11	2_{IV}^{11-6}	$6=123$, $7=124$, $8=134$, $9=125$, $t_0=135$, $t_1=145$ 11th factor	all 11 main effects
11	2_{III}^{11-6}	$6=12$, $7=13$, $8=234$, $9=235$, $t_0=245$, $t_1=1345$	4, 5, 8, 9, t_0, t_1, 14, 15, 18, 19, $1t_0$, $1t_1$

Choice of Fractions and Avoidance of Specific Level Combinations

2^p parallel flats

p independent defining relations: $I = \pm W_1 = \pm W_2 = \cdots = \pm W_p$

- A 2^{k-p} design has 2^p choices. ← isomorphic designs

check LNp.6

- In general, use randomization to choose one of them.

For example, the 2^{7-3} design has 8 choices

$$4 = \pm 12,\ 5 = \pm 13,\ 6 = \pm 23.$$

$2 \times 2 \times 2$

Randomly choose the signs.

different "+" & "−" signs for $W_1, \cdots, W_p$ can generate different choices of the 2^{k-p} design (check LNp.6-6). But, they are isomorphic designs.

- If specific level combinations, e.g.,

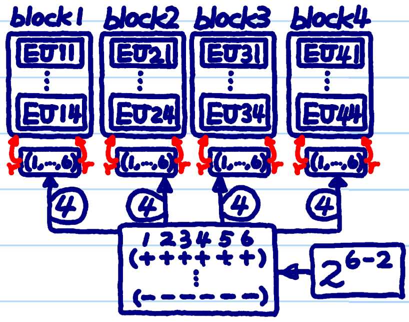
$(+,+,+)$ for high pressure, high temperature, high concentration, ← might cause explosion

are deemed undesirable or even disastrous, they can be avoided by choosing a fraction that does not contain them. Example on p.237 of textbook.

❖ **Reading**: textbook, 5.5

Blocking in FF Designs

← Recall. Arrange full factorial in blocks (LNp. 5-36~40)



16-run FFD → six treatment factors

- Example: Arrange the 2^{6-2} design in four ($= 2^2$) blocks with

one block factor with 4 levels, block size = 4

DCSG for treatment factors → $I = 1235 = 1246 = 3456.$

$5 = 123$ $\quad$ $6 = 124$

- Suppose we choose

2 pseudo block factors

a block ME

$b_1 = 134,\ b_2 = 234,\ b_1b_2 = 12.$

$I = 134b_1$ $\quad$ $I = 234b_2$ $\quad$ $I = 12b_1b_2$

factors {1,2,3,4} form a full factorial

$I = 134b_1 = 234b_2 = 12b_1b_2$ ← DCSG for block factors

- Then

$$b_1 = 134 = 245 = 236 = 156,$$
$$b_2 = 234 = 145 = 136 = 256,$$
$$b_1b_2 = 12 = 35 = 46 = 123456;$$

confounding between treatment and block effects (caused by mixed-type words)

aliasing between treatment effects (caused by pure-type words)

$$13 = 25 = 2346 = 1456,$$
$$14 = 26 = 2345 = 1356,$$
$$15 = 23 = 2456 = 1346,$$
$$16 = 24 = 2356 = 1345,$$
$$34 = 56 = 1245 = 1236,$$
$$36 = 45 = 1256 = 1234.$$

$1 = 235 = 246 = 13456$
$2 = 135 = 146 = 23456$
$3 = 125 = 12346 = 456$
$4 = 12345 = 126 = 356$
$5 = 123 = 12456 = 346$
$6 = 12356 = 124 = 345$

The $4 \times 3 = 12$ factorial effects are confounded with block effects and cannot be used for estimation. Among the remaining 12 degrees of freedom, six are main effects and the rest are given above.

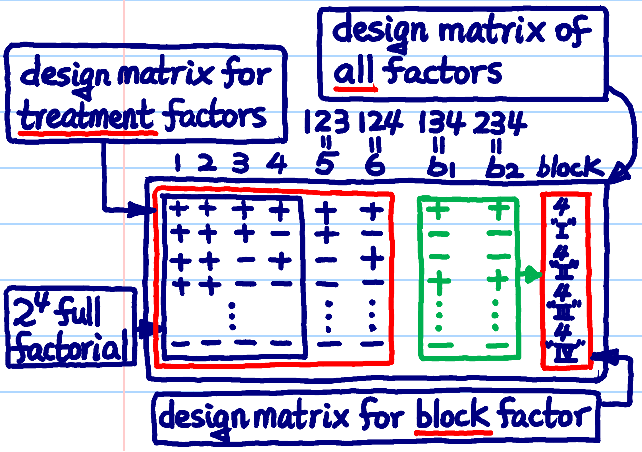
Use of Design Tables for Blocking

- Among the 15 degrees of freedom for the blocked design on LNp.6-30, 3 are allocated for block effects and 6 are for clear main effects (see Table 8 in LNp.6-32). The remaining 6 degrees of freedom are six pairs of aliased two-factor interactions.
- For the 2^{6-2} design with $5 = 12$, $6 = 134$, if we use the block generators $b_1 = 13$, $b_2 = 14$, there are a total of 9 clear effects (see Table 8 in LNp.6-32):

worse aberration than the 2^{6-2} design in LNp.6-30

$$3, 4, 6, 23, 24, 26, 35, 45, 56. \leftarrow \text{(exercise)}$$

  - Thus, the total number of clear effects for this blocked design is 3 more than the total number of clear effects for the blocked design on LNp.6-30.
  - However, only the main effects 3, 4, 6 are clear.



pure-type words

2^{8-4}: $I = 1235 = 1246 = 3456$

mixed-type words

$= 134b_1 = 245b_1 = 236b_1 = 156b_1$
$= 234b_2 = 145b_2 = 136b_2 = 256b_2$
$= 12b_1b_2 = 35b_1b_2 = 46b_1b_2 = 123456b_1b_2$

- alias sets:

$1 = 235 = 246 = 13456$
$= \cancel{34b_1} = \cancel{1245b_1} = \cancel{\cdot\cdot} = \cancel{23456b_1b_2}$

$134 = 245 = 236 = 156 = \boxed{b_1}$
$= \cancel{1235b_1} = \cancel{1246b_1} = \cancel{\cdot\cdot} = \cancel{256b_1b_2}$

Table 8: Sixteen-Run 2^{k-p} Fractional Factorial Designs in 2^q Blocks

of treatment factors

fraction

run size

# of pseudo block factors

k-p=4

# of blocks or # of levels of the block factor

k	p	q	Design Generators	Block Generators	Clear Effects
5	1	1	$5 = 1234$	$b_1 = 12$	all five main effects, all 2fi's except 12
5	1	2	$5 = 1234$	$b_1 = 12$, $b_2 = 13$	all five main effects, 14, 15, 24, 25, 34, 35, 45
5	1	3	$5 = 123$	$b_1 = 14$, $b_2 = 24$, $b_3 = 34$	all five main effects
6	2	1	$5 = 123, 6 = 124$	$b_1 = 134$	all six main effects
6	2	1	$5 = 12, 6 = 134$	$b_1 = 13$	3, 4, 6, 23, 24, 26, 35, 45, 56
6	2	2	$5 = 123, 6 = 124$	$b_1 = 134$, $b_2 = 234$	all six main effects
6	2	2	$5 = 12, 6 = 134$	$b_1 = 13$, $b_2 = 14$	3, 4, 6, 23, 24, 26, 35, 45, 56
6	2	3	$5 = 123, 6 = 124$	$b_1 = 13$, $b_2 = 23$, $b_3 = 14$	all six main effects

2 blocks, block size=8

4 blocks, block size=4

8 blocks, block size=2

cf

cf.

MA

larger # of clear treatment effects

Q: How to generate MA criterion to blocked FFDs?

W_t : WLP of pure-type words
W_b : WLP of mixed-type words

combine into a WLP in a reasonable way, and then sequentially minimize

Table 8: Sixteen-Run 2^{k-p} Fractional Factorial Designs in 2^q Blocks (Cont.)

k-p=4

k	p	q	Design Generators	Block Generators	Clear Effects
7	3	1	5 = 123, 6 = 124, 7 = 134	b_1 = 234	all seven main effects
7	3	2	5 = 123, 6 = 124, 7 = 134	b_1 = 12, b_2 = 13	all seven main effects
7	3	3	5 = 123, 6 = 124, 7 = 134	b_1 = 12, b_2 = 13, b_3 = 14	all seven main effects
8	4	1	5 = 123, 6 = 124, 7 = 134, 8 = 234	b_1 = 12	all eight main effects
8	4	2	5 = 123, 6 = 124, 7 = 134, 8 = 234	b_1 = 12, b_2 = 13	all eight main effects
8	4	3	5 = 123, 6 = 124, 7 = 134, 8 = 234	b_1 = 12, b_2 = 13, b_3 = 14	all eight main effects
9	5	1	5 = 12, 6 = 13, 7 = 14, 8 = 234, 9 = 1234	b_1 = 23	none
9	5	2	5 = 12, 6 = 13, 7 = 14, 8 = 234, 9 = 1234	b_1 = 23, b_2 = 24	none

→ Ok for screening designs.

- More FF designs in blocks are given in Appendix 5B of textbook.

❖ **Reading**: textbook, 5.6