Minimum Aberration Criterion

cf. maximum resolution (LNp.6-10)

- Motivating example: consider the two 2^{7-2} designs:

$$d_1: \quad \mathbf{I} = 4567 = 12346 = 12357,$$

$$d_2: \quad \mathbf{I} = 1236 = 1457 = 234567.$$

causes $\binom{4}{2}=6$ 2fi's aliased in 3 alias sets

  - Both have resolution IV. → 6 2fi's aliased → 6 2fi's aliased (Q: in how many alias sets?) cf.
  - But $W(d_1) = (0,1,2,0,0)$ and $W(d_2) = (0,2,0,1,0)$.

d_3: I=1236=1257=3567 ⇒ 12=36=57=123567

  - Which one is better?
  - Intuitively one would argue that d_1 is better because

$$A_4(d_1) = 1 < A_4(d_2) = 2.$$

(Why? Effect hierarchy principle.)

WLP → $(A_1(d), A_2(d), A_3(d), \cdots, A_k(d))$ sequentially minimize

- For any two 2^{k-p} designs d_1 and d_2, let

$$r = \text{the smallest integer such that } A_r(d_1) \neq A_r(d_2).$$

cf. → MA criterion in blocked full factorial (LNp.5-40)

  - d_1 is said to have *less aberration* than d_2 if $A_r(d_1) < A_r(d_2)$.
  - If no design has less aberration than d_1, then d_1 has *minimum aberration*.

- Throughout the book, this is the *major criterion used for selecting fractional factorial designs*. Its theory is covered in the Mukherjee-Wu (2006) book.

❖ **Reading**: textbook, 5.2

Analysis for Location Effects

Recall. build 2 model (LNp.6-1)

Note. The model matrix X of a 2^{n-k} FFD is identical to that of a 2^{n^*} full factorial, where $n^* = n-k$

except

e.g. I=ABCD $\hat{\beta}_B \to \beta_B + \beta_{ACD}$

- *Same* strategy as in full factorial experiments *except* for the **interpretation and handling of aliased effects**.
- For the location effects (based on $\bar{y}_i$ values), (LNp 6-2) unreplicated
  - the estimated factorial effects are given in Table 3 (LNp.6-14), and
  - the corresponding half-normal plot in Figure 2 (LNp.6-15).
- Visually one may judge that
  - Q, B, C, CQ and possibly E, BQ are significant.

  One can apply the studentized maximum modulus test (see textbook, sec. 4.14, not covered in class) to confirm that Q, B, C, CQ are significant at 0.05 level (see textbook, p.219 and 221).

Use $z = y$ (replicated data, LNp.6-2)
Fit $z = X\beta + \varepsilon$, ε constant var σ^2
Under H_0: $\beta_1 = \cdots = \beta_t = 0$
$\hat{\beta}_1, \cdots, \hat{\beta}_t \overset{i.i.d.}{\sim} N(0, \sigma_*^2)$
$\max_i(|\hat{\beta}_i|)/\hat{\sigma}_*$

- The $B \times Q$ and $C \times Q$ plots (Figure 3, LNp.6-16) show that they are synergystic.
- For illustration, we use the model

$$\frac{\bar{z}(B+) - \bar{z}(B-)}{2}$$

$\in \{-1, +1\}$

$$\hat{y} = 7.6360 + 0.1106x_B + 0.0519x_E + 0.0881x_C - 0.1298x_Q \\ +0.0423x_Bx_Q - 0.0827x_Cx_Q \tag{3}$$

Factorial Effects, Leaf Spring Experiment

Table 3: Factorial Effects, Leaf Spring Experiment

$\boxed{I = BCDE}$

	Effect	$\bar{y}$	$\ln s^2$
CDE =	B	✓ $\boxed{0.221}$	✓ $\boxed{1.891}$
BDE =	C	✓ 0.176	0.569
	D	0.029	−0.247
	E	✓ 0.104	0.216
	Q	✓ −0.260	0.280
	BQ	✓ 0.085	−0.589
	CQ	✓ −0.165	0.598
	DQ	0.054	✓ 1.111
	EQ	0.027	0.129
	BC	0.017	−0.002
	BD	0.020	0.425
	CD	−0.035	0.670
	BCQ	0.010	✓ −1.089
	BDQ	−0.040	−0.432
	BEQ	−0.047	0.854

$\bar{z}(B+) - \bar{z}(B-)$

$= \bar{z}(CDE+) - \bar{z}(CDE-)$

$= 2\hat{\beta}_{B=CDE}$

Half-normal Plot of Location Effects, Leaf Spring Experiment

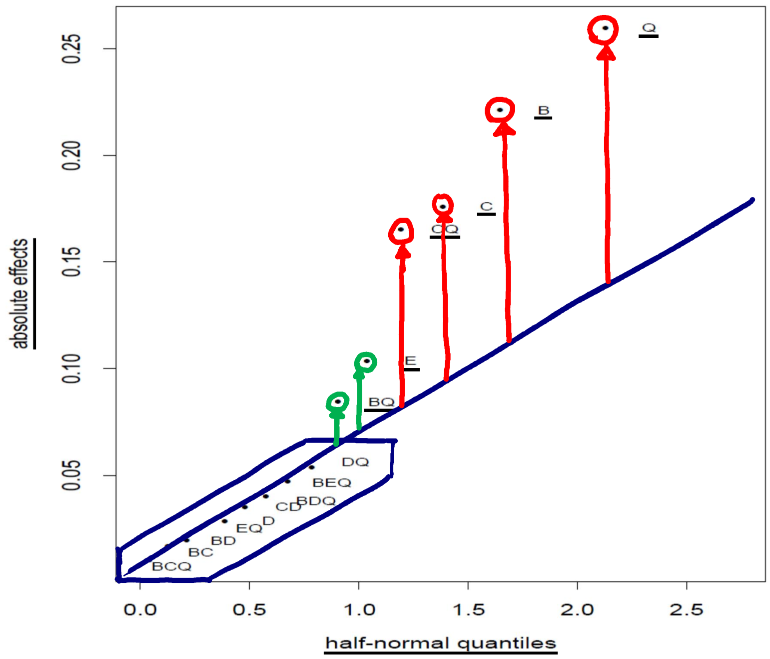


Figure 2: Half-Normal Plot of Location Effects, Leaf Spring Experiment

(exercise) Apply Lenth's method

Interaction Plots

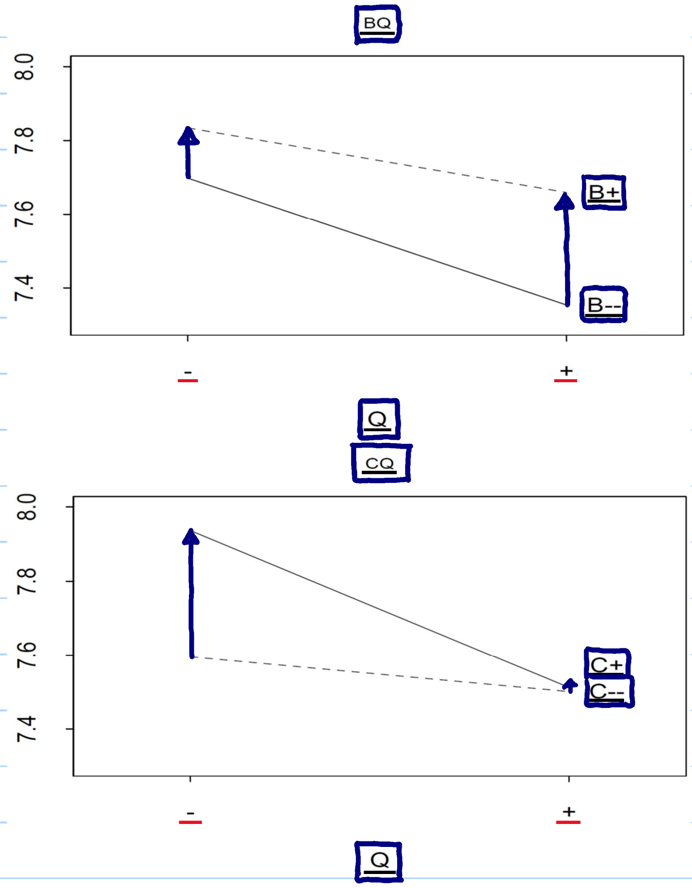


Figure 3: $B \times Q$ and $C \times Q$ interaction plots, Leaf Spring Experiment

Analysis for Dispersion Effects

I=BCDE

- For the dispersion effects (based on $z_i = \ln s_i^2$ values), (LNp.6-2) unreplicated
  - the estimated factorial effects are given in Table 3 (LNp.6-14)
  - the half-normal plot is given in Figure 4 (LNp.6-18).
- Visually only effect B stands out. This is confirmed by applying the studentized maximum modulus test (see textbook, sec. 4.14).
- For illustration, we will include B, DQ, BCQ in the following model,

I=BCDE

$$\ln \hat{\sigma}^2 = -4.9313 + \underbrace{0.9455}_{\beta_B+\beta_{CDE}} x_B + \underbrace{0.5556}_{\beta_{DQ}+\beta_{BCEQ}} x_D x_Q - \underbrace{0.5445}_{\beta_{BCQ}+\beta_{DEQ}} x_B x_C x_Q. \qquad (4)$$

(effect hierarchy) ≈0 ≈0 (effect hierarchy) ≈0 (effect heredity)

- The $D \times Q$ and $B \times C \times Q$ interaction plots are given in Figures 5 and 6 (LNp.6-19).

Q: What settings of (B,C,D,Q) can minimize $\ln\hat{\sigma}^2$ using (4)?

B	C	D	Q	DQ	BCQ
−	+	+	+	+	−
	+	+	−	−	+
	+	−	+	−	−
	+	−	−	+	+
	−	+	+	+	+
	−	+	−	−	−
	−	−	+	−	+
	−	−	−	+	−

$\xleftrightarrow{\text{cf.}}$ step 1 in LNp.6-20

Half-normal Plot of Dispersion Effects, Leaf Spring Experiment

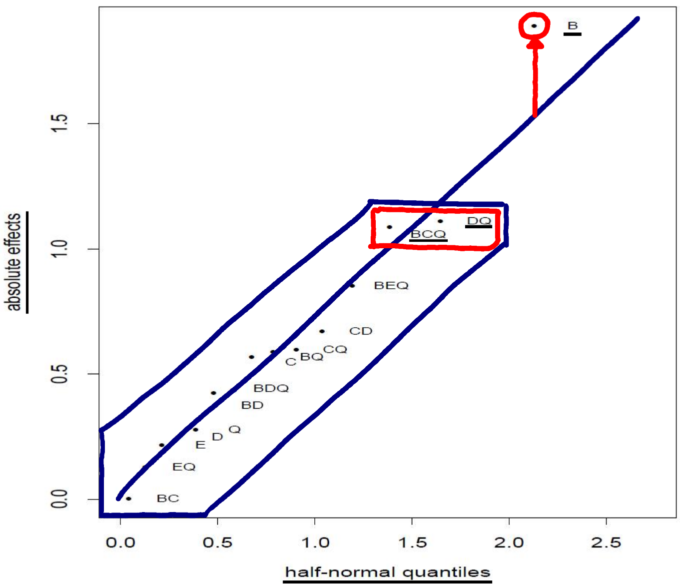


Figure 4: Half-Normal Plot of Dispersion Effects, Leaf Spring Experiment

(exercise) Apply Lenth's method

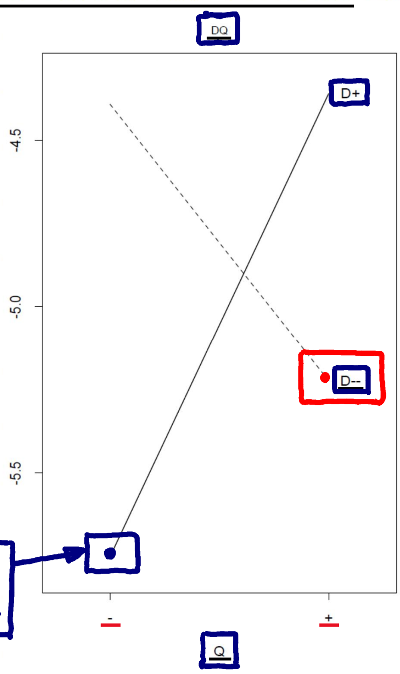
Interaction Plots for Dispersion Effects

(LNp.6-17)

B C D Q
− + + −

B C D Q
− − − +

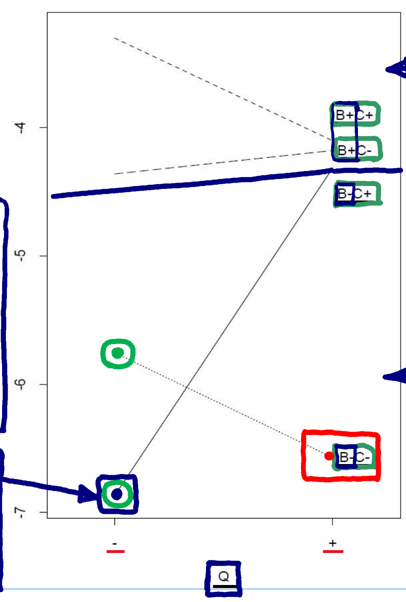
Q: Why the 2 settings are different in the interaction plots?



D=+
Q=−

Figure 5 : $D \times Q$ Interaction Plot

BCQ



B=−
C=+
Q=−

Similar 2-factor interaction plots or not?

Figure 6 : $B \times C \times Q$ Interaction Plot

Fitted model in LNp.6-17 does not contain ME D & ME Q.

Check the definition of 3-factor interaction in LNp.5-8

Two-Step Procedure for Optimization

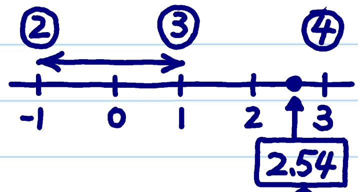
- Step 1: To minimize s^2 (or $\ln s^2$), we can
  - choose $B = -$ based on eq. (4) in LNp.6-17,
  - choose the combination with the lowest value, $D = +$, $Q = -$ based on the $D \times Q$ plot (Figure 5, LNp.6-19),
  - with $B = -$ and $Q = -$, choose $C = +$ to attain the minimum in the $B \times C \times Q$ interaction plot (Figure 6, LNp.6-19).

Another confirmation: they lead to $x_B = -, x_D x_Q = -$ and $x_B x_C x_Q = +$ in the model (4), which make each of the last three terms negative. ← check LNp.6-17

- Step 2: With $(B,C,D,Q) = (-,+,+,-)$, from model (3) in LNp.6-13, we have

$$\begin{aligned}\hat{y} &= 7.6360 + 0.1106(-1) + 0.0519x_E + 0.0881(+1) - 0.1298(-1)\\ &\quad +0.0423(-1)(-1) - 0.0827(+1)(-1)\\ &= 7.8683 + 0.0519x_E.\end{aligned}$$

← adjustment factor



By solving $\hat{y} = 8.0$, $x_E = 2.54$.

Warning: This is way outside the experimental range for factor E. Such a value may not make physical sense and the predicted variance value for this setting may be too optimistic and not substantiated.

extrapolation

follow-up confirm experiment.

❖ **Reading**: textbook, 5.3