design matrix of 2^4 full factorial

Four 2-level factors

model matrix of 2^4 with all factorial effects

$(abcd)*(cd) = I = abc^2d^2 = ab$

run	A	B	C	D	a	b	c	d	ab	ac	ad	bc	bd	cd	abc	abd	acd	bcd	abcd	I
1	–1	–1	–1	–1	–1	–1	–1	–1	1	1	1	1	1	1	–1	–1	–1	–1	1	1
2	–1	–1	–1	1	–1	–1	–1	1	1	1	–1	1	–1	–1	–1	1	1	1	–1	1
3	–1	–1	1	–1	–1	–1	1	–1	1	–1	1	–1	1	–1	1	–1	1	1	–1	1
4	–1	–1	1	1	–1	–1	1	1	1	–1	–1	–1	–1	1	1	1	–1	–1	1	1
5	–1	1	–1	–1	–1	1	–1	–1	–1	1	1	–1	–1	1	1	1	–1	1	–1	1
6	–1	1	–1	1	–1	1	–1	1	–1	1	–1	–1	1	–1	1	–1	1	–1	1	1
7	–1	1	1	–1	–1	1	1	–1	–1	–1	1	1	–1	–1	–1	1	1	–1	1	1
8	–1	1	1	1	–1	1	1	1	–1	–1	–1	1	1	1	–1	–1	–1	1	–1	1
9	1	–1	–1	–1	1	–1	–1	–1	–1	–1	–1	1	1	1	1	1	1	–1	–1	1
10	1	–1	–1	1	1	–1	–1	1	–1	–1	1	1	–1	–1	1	–1	–1	1	1	1
11	1	–1	1	–1	1	–1	1	–1	–1	1	–1	–1	1	–1	–1	1	–1	1	1	1
12	1	–1	1	1	1	–1	1	1	–1	1	1	–1	–1	1	–1	–1	1	–1	–1	1
13	1	1	–1	–1	1	1	–1	–1	1	–1	–1	–1	–1	1	–1	–1	1	1	1	1
14	1	1	–1	1	1	1	–1	1	1	–1	1	–1	1	–1	–1	1	–1	–1	–1	1
15	1	1	1	–1	1	1	1	–1	1	1	–1	1	–1	–1	1	–1	–1	–1	–1	1
16	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

How to obtain a $2^{4-2} = 2^4/2^2$ FFD?

$I = abcd$ & $I = cd$

design matrix of the 2^{4-2} {1, 4, 13, 16}

4 runs ⇒ 4 df ⇒ can etimate 3 (joint) effects

Which 3?

defining contrast subgroup → $I = abcd = cd = ab$ ← defining words

$a = a^2bcd = acd = a^2b$ ← It estimates the parameter $\beta_a + \beta_{bcd} + \beta_{acd} + \beta_b$

$a^2bcd = bcd$, $a^2b = b$

$c = abd = d = abc$

$ac = bd = ad = bc$

In runs {1, 4, 13, 16}, factors A & C form a 2^2 full factorial

Clear Effects

How to detect? The information is in the alias sets

- A main effect or two-factor interaction (2fi) is called **clear** if it is not aliased with any other m.e.'s or 2fi's and **strongly clear** if it is not aliased with any other m.e.'s, 2fi's or 3fi's. Therefore a clear effect is *estimable* under the assumption of negligible 3-factor and higher-order interactions and a strongly clear effect is *estimable* under the weaker assumption of negligible 4-factor and higher-order interactions.

$\beta_{ABC}=0$, ⋮, $\beta_{ABCD}=0$, ⋮

$\beta_{ABCD}=0, \cdots, \beta_{ABCDE}=0, \cdots, \cdots$

Intuition. For important effects, it would be better for them to be aliased with higher-order interactions (Why?) ← effect hierarchy principle

- In the 2^{5-1} design with $\mathbf{I} = BCDE$, which effects are clear and strongly clear?

$I = BCD$ cf. cf.

check LNp6-4

Ans: B, C, D, E are clear, Q, BQ, CQ, DQ, EQ are strongly clear.

- Consider the alternative plan 2^{5-1} design with $\mathbf{I} = BCDEQ$. (It is said to have resolution V because the length of the defining word is 5 while the previous plan has resolution IV.) It can be verified that all five main effects are strongly clear and all 10 2fi's are clear. (*Do the derivations*). This is a very good plan because *each* of the 15 degrees of freedom is either clear or strongly clear.

# of letters (factors) involved in a word

Intuition. It would be better to have longer words in DCSG.

DCSG → Defining Contrast Subgroup for 2^{k-p} Designs

can be used to
① construct design matrix
② study effect aliasing
③ develop criteria for choosing "good" designs

df. for effect (alias set) estimation

- A 2^{k-p} design has k factors, 2^{k-p} runs, and it is a 2^{-p}th fraction of the 2^k design. The fraction is defined by p *independent* defining words. → $W_1, \cdots, W_p$

check LNp.6-6. What can be the 3rd word?

group defined in Algebra

The group formed by these p words is called the **defining contrast subgroup**. It has $2^p - 1$ words plus the identity element **I**. → $\{I, W_1, \cdots, W_p, W_1W_2, \cdots, W_{p-1}W_p, W_1W_2W_3, \cdots, W_1W_2\cdots W_p\}$ generators

- **Resolution** = shortest wordlength among the $2^p - 1$ words. # of words in the group $= 2^p$ Why?

6 factors 1,2,⋯,6

2 generators → generate $2^2 = 4$ words (including I)

- Example: A 2^{6-2} design with $\mathbf{5} = \mathbf{12}$ and $\mathbf{6} = \mathbf{134}$. The two independent defining words are $\mathbf{I} = \mathbf{125}$ and $\mathbf{I} = \mathbf{1346}$. Then $\mathbf{I} = \mathbf{125} \times \mathbf{1346} = \mathbf{23456}$. The defining contrast subgroup $= \{\mathbf{I}, \mathbf{125}, \mathbf{1346}, \mathbf{23456}\}$. The design has resolution III.

$2^4 = 16$ runs

	WL
125	3
1346	4
23456	5

a column: an alias set

model matrix

a group (Algebra)

16 runs 2^4 full factorial

1	2	3	4	12 (= 5)	13	14	23	⋯	34	123	⋯	134 (= 6)	⋯	234	1234	I
+	+	+	+	+	+	+	+					+			+	+
+	+	+	−	+	+	−	+					−			−	+
+	+	−	+	+	−	+	−					−			−	+
+	−	+	+	−	+	+	−					+			−	+
−	+	+	+	−	−	−	+					−			−	+
⋮				⋮	⋮	⋮	⋮					⋮			⋮	⋮
−	−	−	−	+	+	+	+					−			+	+

2^{6-2} design matrix

$1+2+5=0 \pmod 2$
$1+3+4+6=0 \pmod 2$
$2+3+4+5+6=0 \pmod 2$
where $1,2,3,4,5,6 \in \{0,1\}$

Deriving Aliasing Relations for the 2^{6-2} design

$= 2^4$

- For the same 2^{6-2} design, the defining contrast subgroup is

word (defined on design matrix)

$$\mathbf{I} = \mathbf{125} = \mathbf{1346} = \mathbf{23456}. \qquad (1)$$

$125 \times 1346 = 23456$

most serious aliasing?
1=25
2=15
5=12

most serious aliasing?
13=46
14=36
16=34

I	=	125	=	1346	=	23456,
✗1	=	✗25	=	346	=	123456,
✗2	=	✗15	=	12346	=	3456,
✓3	=	1235	=	146	=	2456,
✓4	=	1245	=	136	=	2356,
✗5	=	✗12	=	13456	=	2346,
✓6	=	1256	=	134	=	2345,
✗13	=	235	=	✗46	=	12456,
✗14	=	245	=	✗36	=	12356,
✗16	=	256	=	✗34	=	12345,
✓23	=	135	=	1246	=	456,
✓24	=	145	=	1236	=	356,
✓26	=	156	=	1234	=	345,
✓35	=	123	=	1456	=	246,
✓45	=	124	=	1356	=	236,
✓56	=	126	=	1345	=	234.

same as the defining contrast subgroup.

there are 16 alias sets

5=12 in LNp.6-8 cf.

an alias set

effect (appears in model matrix)

factors {1,2,3,4} forms a full factorial

{1,3,4,6} not form a full factorial

All the 15 degrees of freedom (each is a coset in group theory) are identified.

- It has the clear effects: **3, 4, 6, 23, 24, 26, 35, 45, 56**. It has resolution III.

WordLength Pattern and Resolution

↳ WLP ← defined on defining contrast subgroup

- Define A_i = number of defining words of length i.
- $W = (A_3, A_4, A_5, \ldots)$ is called the **wordlength pattern**. ←cf.→ WLP of block scheme (LNp 5-40)
  - In this design, $W = (1, 1, 1, 0)$.
  - It is required that $A_1 = 0$ and $A_2 = 0$. (Why? No main effect is allowed to be aliased with the intercept or another main effect.)
- **Resolution** = smallest r such that $A_r \geq 1$. ← resolution is a function of WLP
- **Maximum resolution criterion**: For fixed k and p, choose a 2^{k-p} design with maximum resolution.
- **Rules for Resolution IV and V Designs:**

It is better to have higher resolution

(i) In any resolution IV design, the main effects are clear.

resolution IV
e.g. I=ABCD
AB=CD
AC=BD
AD=BC

(ii) In any resolution V design, the main effects are strongly clear and the two-factor interactions are clear. (2)

resolution V
e.g. I=ABCDE
AB=CDE , BD=ACE
AC=BDE , BE=ACD
AD=BCE , CD=ABE
AE=BCD , CE=ABD
BC=ADE , DE=ABC

(iii) Among the resolution IV designs with given k and p, those with the largest number of clear two-factor interactions are the best.

A Projective Rationale for Resolution

(✱) A constraint (e.g., I = a word) on a submatrix must be a constraint on the whole design matrix.

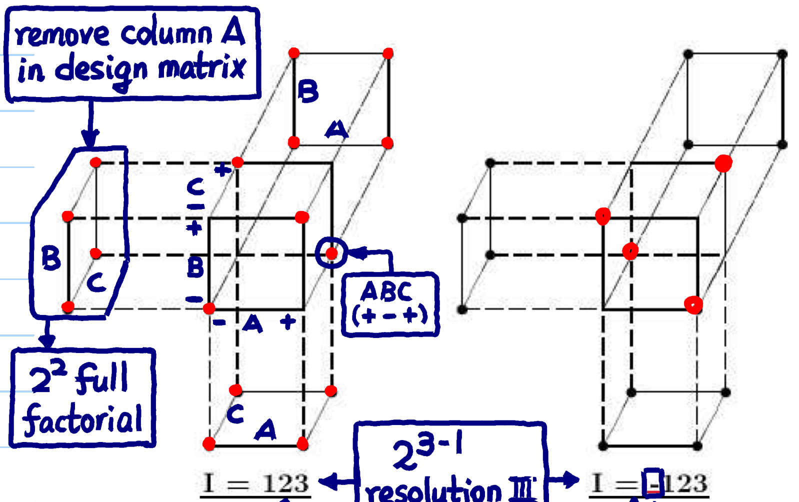
projection of design, 2 viewpoints:
① matrix viewpoint ⟶ submatrix of design matrix
② geometric viewpoint

Why?

- For a resolution R design, its projection onto any $R-1$ factors is a full factorial in the $R-1$ factors. This would allow *effects of all orders among the $R-1$ factors to be estimable*. (**Caveat**: it makes the assumption that other factors are inert.) ← cf.

By (✱) "Some effects are inert" (effect hierarchy or effect sparsity)

e.g. 2^{7-2} with I=1236=1457=23457
projected onto {1,2,3,4} ⇒ 2^4 full, 2 replicates
projected onto {1,2,3,4,5,6} ⇒ 2^{6-1} with I=1236
(exercise) {1,2,3,6} {2,4,5,6,7}



$1+2+3 = 0 \pmod 2$ cf. $1+2+3 = 1 \pmod 2$ cf. Check LNp 6-5 → 1=−23, 2=−13, 3=−12

Q: Why is this property useful? Suppose that only R-1 or fewer "factors" are important, ⇒ all effects of these important factors are located in different alias sets.

Figure 1: 2^{3-1} Designs Using $\mathbf{I} = \pm\mathbf{123}$ and Their Projections to 2^2 Designs.

Minimum Aberration Criterion

$\xleftrightarrow{\text{cf}}$ maximum resolution (LNp.6-10)

- Motivating example: consider the two 2^{7-2} designs:

$$d_1: \quad \mathbf{I} = 4567 = 12346 = 12357,$$

$$d_2: \quad \mathbf{I} = 1236 = 1457 = 234567.$$

causes $\binom{4}{2}=6$ 2fi's aliased in 3 alias sets

  - Both have resolution IV. → 6 2fi's aliased → 6 2fi's aliased (Q: in how many alias sets?)

  cf d_3: I=1236=1257=3567 ⇒ 12=36=57=123567

  - But $W(d_1) = (0,1,2,0,0)$ and $W(d_2) = (0,2,0,1,0)$.
  - Which one is better?
  - Intuitively one would argue that d_1 is better because
    $$A_4(d_1) = 1 < A_4(d_2) = 2.$$
    (Why? Effect hierarchy principle.)

WLP → $(A_1(d), A_2(d), A_3(d), \cdots, A_k(d))$ sequentially minimize

- For any two 2^{k-p} designs d_1 and d_2, let
  $$r = \text{the smallest integer such that } A_r(d_1) \neq A_r(d_2).$$

  cf → MA criterion in blocked full factorial (LNp.5-40)

  - d_1 is said to have *less aberration* than d_2 if $A_r(d_1) < A_r(d_2)$.
  - If no design has less aberration than d_1, then d_1 has *minimum aberration*.
- Throughout the book, this is the *major criterion used for selecting fractional factorial designs*. Its theory is covered in the Mukherjee-Wu (2006) book.

❖ **Reading**: textbook, 5.2

Analysis for Location Effects

Recall build 2 model (LNp.6-1)

Note. The model matrix X of a 2^{n-k} FFD is identical to that of a 2^{n^*} full factorial, where $n^* = n-k$

except

e.g. I=ABCD
$\hat{\beta}_B \to \widehat{\beta_B + \beta_{ACD}}$

- *Same* strategy as in full factorial experiments *except* for the **interpretation and handling of aliased effects**.
- For the location effects (based on $\bar{y}_i$ values), (LNp 6-2) unreplicated
  - the estimated factorial effects are given in Table 3 (LNp.6-14), and
  - the corresponding half-normal plot in Figure 2 (LNp.6-15).
- Visually one may judge that
  - Q, B, C, CQ and possibly E, BQ are significant.

  One can apply the studentized maximum modulus test (see textbook, sec. 4.14, not covered in class) to confirm that Q, B, C, CQ are significant at 0.05 level (see textbook, p.219 and 221).

Use $z = y$ (replicated data, LNp 6-2)
Fit $z = X\beta + \epsilon$ (ϵ: constant var σ^2)
Under H_0: $\beta_1 = \cdots = \beta_t = 0$
$\hat{\beta}_1, \cdots, \hat{\beta}_t \overset{\text{i.i.d.}}{\sim} N(0, \sigma_*^2)$
$\max_i(|\hat{\beta}_i|)/\hat{\sigma}_*$

- The $B \times Q$ and $C \times Q$ plots (Figure 3, LNp.6-16) show that they are synergystic.
- For illustration, we use the model

$\dfrac{\bar{z}(B+) - \bar{z}(B-)}{2}$ $\quad \in \{-1, +1\}$

$$\hat{y} = 7.6360 + 0.1106x_B + 0.0519x_E + 0.0881x_C - 0.1298x_Q + 0.0423x_Bx_Q - 0.0827x_Cx_Q \tag{3}$$

Factorial Effects, Leaf Spring Experiment

Table 3: Factorial Effects, Leaf Spring Experiment

$\boxed{I = BCDE}$

	Effect	$\bar{y}$	$\ln s^2$
CDE =	B	✓ $\boxed{0.221}$	✓ $\boxed{1.891}$
BDE =	C	✓ 0.176	0.569
	D	0.029	−0.247
	E	✓ 0.104	0.216
·	Q	✓ −0.260	0.280
·	BQ	✓ 0.085	−0.589
·	CQ	✓ −0.165	0.598
	DQ	0.054	✓ 1.111
	EQ	0.027	0.129
	BC	0.017	−0.002
	BD	0.020	0.425
	CD	−0.035	0.670
	BCQ	0.010	✓ −1.089
	BDQ	−0.040	−0.432
	BEQ	−0.047	0.854

$\bar{\bar{z}}(B+) - \bar{\bar{z}}(B-)$
$= \bar{\bar{z}}(CDE+) - \bar{\bar{z}}(CDE-)$
$= 2\hat{\beta}_{B=CDE}$

Half-normal Plot of Location Effects, Leaf Spring Experiment

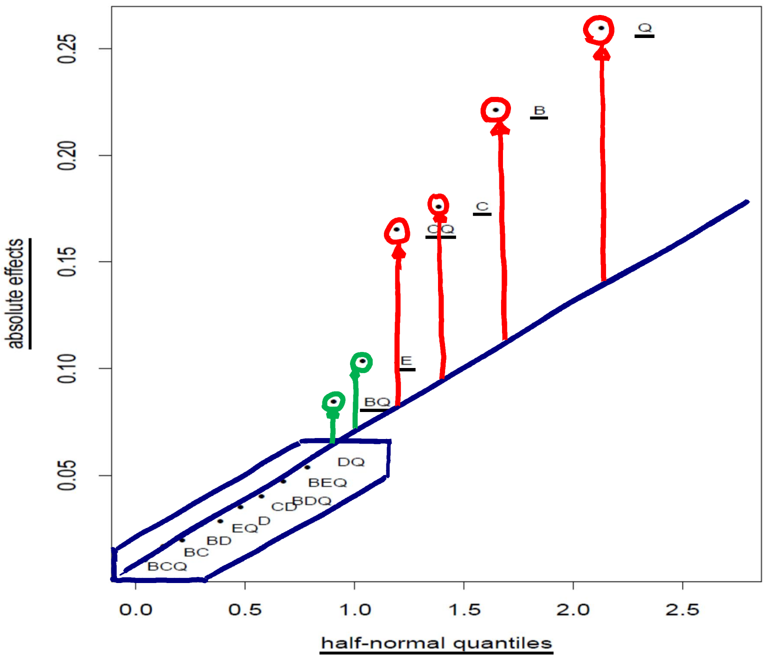


Figure 2: Half-Normal Plot of Location Effects, Leaf Spring Experiment

(exercise) Apply Lenth's method

Interaction Plots

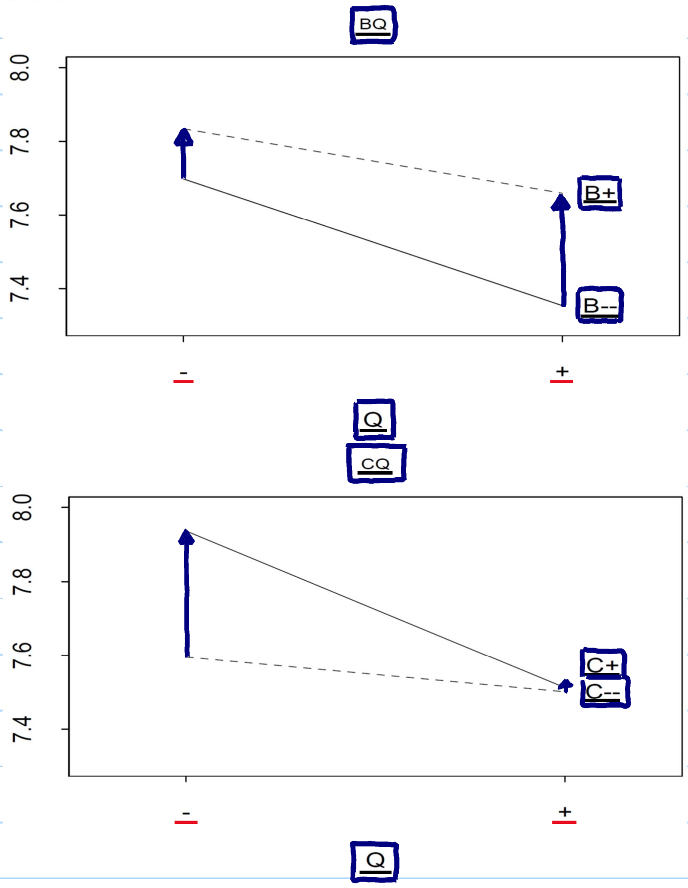


Figure 3: $B \times Q$ and $C \times Q$ interaction plots, Leaf Spring Experiment

Analysis for Dispersion Effects $\boxed{I = BCDE}$

- For the dispersion effects (based on $z_i = \ln s_i^2$ values), (LNp 6-2) unreplicated
  - the estimated factorial effects are given in Table 3 (LNp.6-14)
  - the half-normal plot is given in Figure 4 (LNp.6-18).
- Visually only effect B stands out. This is confirmed by applying the studentized maximum modulus test (see textbook, sec. 4.14).
- For illustration, we will include B, DQ, BCQ in the following model,

$$\ln \hat{\sigma}^2 = \underbrace{-4.9313}_{} + \underbrace{0.9455 x_B}_{B_B + B_{CDE}} + \underbrace{0.5556 x_D x_Q}_{B_{DQ} + B_{BCEQ}} - \underbrace{0.5445 x_B x_C x_Q}_{B_{BCQ} + B_{DEQ}}. \quad (4)$$

(effect hierarchy) ≈ 0 ≈ 0 (effect hierarchy) ≈ 0 (effect heredity)

- The $D \times Q$ and $B \times C \times Q$ interaction plots are given in Figures 5 and 6 (LNp.6-19).

Q: What settings of (B,C,D,Q) can minimize $\ln \hat{\sigma}^2$ using (4)?

B	C	D	Q	DQ	BCQ
−	+	+	+	+	−
	+	+	−	−	+
	+	−	+	−	−
	+	−	−	+	+
	−	+	+	+	+
	−	+	−	−	−
	−	−	+	−	+
	−	−	−	+	−

$\xleftrightarrow{cf.}$ step 1 in LNp 6-20

Half-normal Plot of Dispersion Effects, Leaf Spring Experiment

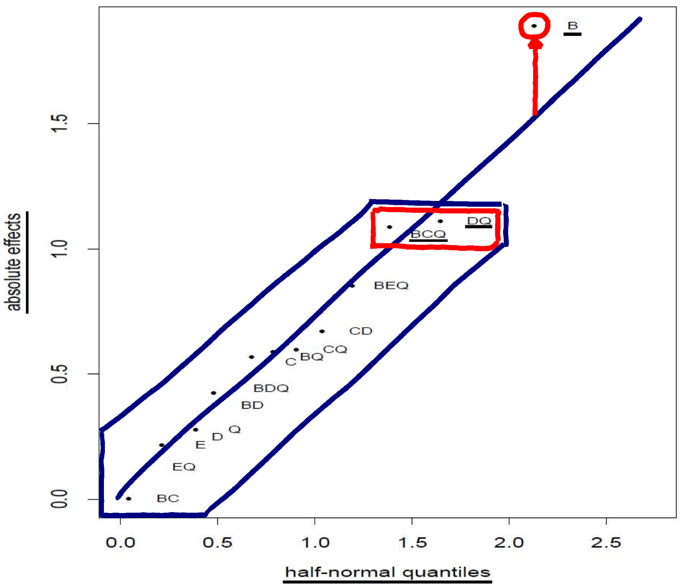


Figure 4: Half-Normal Plot of Dispersion Effects, Leaf Spring Experiment

(exercise) Apply Lenth's method

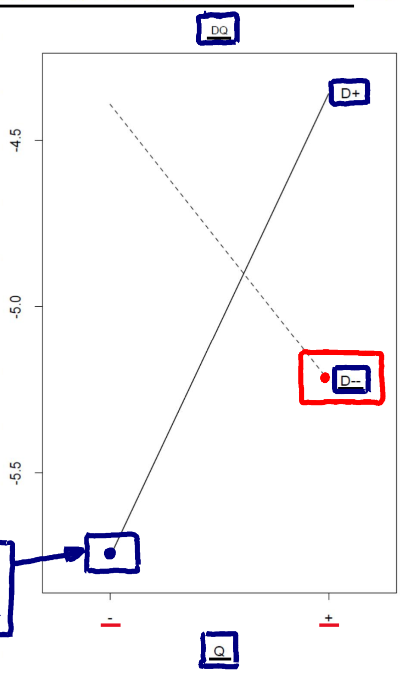
Interaction Plots for Dispersion Effects

(LNp.6-17)

B C D Q
− + + −

B C D Q
− − − +

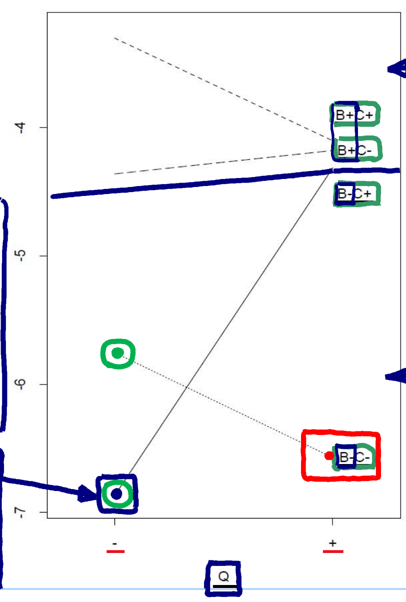
Q: Why the 2 settings are different in the interaction plots?



D=+
Q=−

B=−
C=+
Q=−

BCQ



similar 2-factor interaction plots or not?

Figure 5 : $D \times Q$ Interaction Plot

Figure 6 : $B \times C \times Q$ Interaction Plot

Fitted model in LNp.6-17 does not contain ME D & ME Q.

Check the definition of 3-factor interaction in LNp.5-8

Two-Step Procedure for Optimization

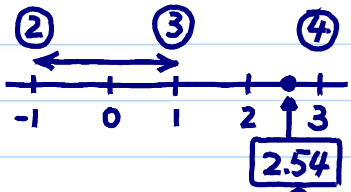
- Step 1: To minimize s^2 (or $\ln s^2$), we can
  - choose $B = -$ based on eq. (4) in LNp.6-17,
  - choose the combination with the lowest value, $D = +$, $Q = -$ based on the $D \times Q$ plot (Figure 5, LNp.6-19),
  - with $B = -$ and $Q = -$, choose $C = +$ to attain the minimum in the $B \times C \times Q$ interaction plot (Figure 6, LNp.6-19).

Another confirmation: they lead to $x_B = -, x_D x_Q = -$ and $x_B x_C x_Q = +$ in the model (4), which make each of the last three terms negative. ← check LNp 6-17

- Step 2: With $(B, C, D, Q) = (-, +, +, -)$, from model (3) in LNp.6-13, we have

$$
\begin{aligned}
\hat{y} &= 7.6360 + 0.1106(-1) + 0.0519 x_E + 0.0881(+1) - 0.1298(-1) \\
&\quad + 0.0423(-1)(-1) - 0.0827(+1)(-1) \\
&= 7.8683 + 0.0519 x_E .
\end{aligned}
$$

← adjustment factor



By solving $\hat{y} = 8.0$, $x_E = 2.54$.

Warning: This is way outside the experimental range for factor E. Such a value may not make physical sense and the predicted variance value for this setting may be too optimistic and not substantiated.

extrapolation

follow-up confirm experiment

❖ **Reading**: textbook, 5.3