Leaf Spring Experiment

* response : free height
* treatment factors : B, C, D, E, Q
  all 2 levels — −1, +1
* Exp'tal units : a spring
  homogeneous — EU1 ⋯ EU48

- Five factors at two levels, use a 16-run design with three replicates for each run. It is a 2^{5-1} design, 1/2 fraction of the 2^5 design.

full factorial 2^5 : # of all level combinations $= 2 \times 2 \times 2 \times 2 \times 2 = 2^5 = 32$

2^{n-k} design: $= 2^n/2^k$
① n: # of factors ② 2: 2 levels ③ k: # of independent defining words ④ 2^{n-k}: run size ⑤ 2^{-k}: fraction

Q. There are enough EUs (48) for 2^5 design (32). Why not do 2^5?

2^{5-1} design repeat 3 times

2^5 design 16

Hint. 2 types of df.

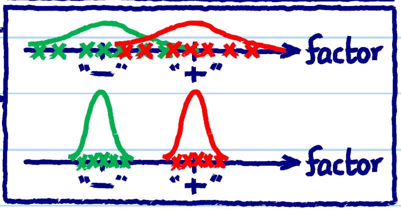
Table 1: Factors and Levels, Leaf Spring Experiment

qualitative or quantitative ?

	Factor	Level −	Level +
heating stage	*B*. high heat temperature (°F)	1840	1880
	C. heating time (seconds)	23	25
forming stage	*D*. transfer time (seconds)	10	12
	E. hold down time (seconds)	2	3
quenching stage	*Q*. quench oil temperature (°F)	130-150	150-170

* conceptual model :
  2^5: $Z \sim \beta_0 + \text{all factorial effects} + \varepsilon$ (31)
  2^{5-1}: $Z \sim \beta_0 + \text{some factorial effects} + \varepsilon$ (15)

measurement errors/errors in predictors (LM, LNp. 9-7)



- response y = free height of spring, target = 8.0 inches.

Goal : get y as close to 8.0 as possible (nominal-the-best problem). → build 2 models:
one for $\mu_x = E(y_x)$ ⟷ factors
one for $\sigma_x^2 = Var(y_x)$ ⟷ factors

Leaf Spring Experiment: Design Matrix and Data

Q: Which 16 level combinations should we choose out of the 32 ones?

Only contains 16 out of all 32 level combinations ⇒ It is called a Fractional Factorial Design (FFD) ⇒ can estimate 15 factorial effects

Q: Which 15 effects?

Table 2: Design Matrix and Free Height Data, Leaf Spring Experiment

B	*C*	*D*	BCD	*E*	*Q*	Free Height (y) (replicated)			$\bar{y}_i$	s_i^2	$\ln s_i^2$
−	+	+	−	−	−	7.78	7.78	7.81	7.7900	0.0003	−8.1117
+	+	+	+	+	−	8.15	8.18	7.88	8.0700	0.0273	−3.6009
−	−	+	+	+	−	7.50	7.56	7.50	7.5200	0.0012	−6.7254
+	−	+	−	−	−	7.59	7.56	7.75	7.6333	0.0104	−4.5627
−	+	−	+	+	−	7.94	8.00	7.88	7.9400	0.0036	−5.6268
+	+	−	−	−	−	7.69	8.09	8.06	7.9467	0.0496	−3.0031
−	−	−	−	−	−	7.56	7.62	7.44	7.5400	0.0084	−4.7795
+	−	−	+	+	−	7.56	7.81	7.69	7.6867	0.0156	−4.1583
−	+	+	−	−	+	7.50	7.25	7.12	7.2900	0.0373	−3.2888
+	+	+	+	+	+	7.88	7.88	7.44	7.7333	0.0645	−2.7406
−	−	+	+	+	+	7.50	7.56	7.50	7.5200	0.0012	−6.7254
+	−	+	−	−	+	7.63	7.75	7.56	7.6467	0.0092	−4.6849
−	+	−	+	+	+	7.32	7.44	7.44	7.4000	0.0048	−5.3391
+	+	−	−	−	+	7.56	7.69	7.62	7.6233	0.0042	−5.4648
−	−	−	−	−	+	7.18	7.18	7.25	7.2033	0.0016	−6.4171
+	−	−	+	+	+	7.81	7.50	7.59	7.6333	0.0254	−3.6717

($\bar{y}_i$, s_i^2, $\ln s_i^2$: unreplicated)

Q: What is the model matrix of this FFD?

Under a conceptual model $\xrightarrow{\text{codings}}$ model matrix X

for replicated response : $y = X\beta^* + \varepsilon^*$

for unreplicate response : $\bar{y} = X\beta_1 + \varepsilon_1$

$\ln S^2 = X\beta_2 + \varepsilon_2$

Why Use Fractional Factorial Designs (FFDs)?

$2^5-1=$

- If a 2^5 design is used for the experiment, its 31 degrees of freedom would be allocated as follows:

	Main Effects	Interactions			
		2-Factor	3-Factor	4-Factor	5-Factor
#	5	$\binom{5}{2}=10$	$\binom{5}{3}=10$	$\binom{5}{4}=5$	$\binom{5}{5}=1$

more important 15 16 less important

- Using effect hierarchy principle, one would argue that 4fi's, 5fi and even 3fi's are not likely to be important. There are 10+5+1 = 16 (16/32 = 1/2) such effects, half of the total runs! Using a 2^5 design can be wasteful (unless 32 runs cost about the same as 16 runs.) cf. $2^{5-1}=16$ runs → 15 effects
- Use of an FFD instead of full factorial design is usually done for economic reasons. Since there is **no free lunch**, what **price to pay**? See next slide.

★ one of the most important concepts in DOE

❖ **Reading**: textbook, 5.1

Effect Aliasing and Defining Relation

check LNp.6-2

- In the design matrix, col $B\times$ col $C\times$ col D = col E. That means,

It estimate $2(\beta_E+\beta_{BCD})$

$$2\hat\beta_E = ME(E) = \bar y(E+) - \bar y(E-) = \bar y(BCD+) - \bar y(BCD-) = INT(BCD) = 2\hat\beta_{BCD}$$

Therefore the design is not capable of distinguishing E from BCD. The main effect E is **aliased** with the interaction BCD. Notationally,

defined on model matrix → $E = BCD$ or $\mathbf{I} = BCDE$, ← Note. It is defined on design matrix

$E^2 = BCDE$ → $B+C+D+E=0 \pmod 2$, where $B,C,D,E\in\{0,1\}$ (exercise) group of additivity

where $\mathbf{I}$ (= column of +'s) is the identity element in the group of multiplications. group of multiplication ← $\{+1,-1\}$

Intercept in the model matrix

(Notice the mathematical similarity between aliasing and confounding. What is the difference?) check LNp.5-36

btwn treatment & block effects | btwn treatment effects

# of all effects (including I): $2^n=2^5=32$ effects
In general, 2^{n-k}: 16 × 2 : In general, 2^k
# of alias sets | # of effects in an alias set

- $\mathbf{I} = BCDE$ is the **defining relation** for the 2^{5-1} design. → $2^4=16$ runs → 15 df for effect estimation → used on 15 alias sets

It implies all the 15 effect aliasing relations :

an alias set ← $\boxed{B = CDE}$, $C = BDE$, $D = BCE$, $E = BCD$,
$BC = DE$, $BD = CE$, $BE = CD$,
$Q = BCDEQ$, $BQ = CDEQ$, $CQ = BDEQ$, $DQ = BCEQ$,
$EQ = BCDQ$, $BCQ = DEQ$, $BDQ = CEQ$, $BEQ = CDQ$.

effects formed by factors B, C, D, Q

In the model matrix of 2^{n-k}, each column corresponds to an alias set, & columns of different alias sets are orthogonal.

design matrix of a full factorial 2^3

Three 2-level factors

model matrix of 2^3 with all factorial effects

	A	B	C	a	b	c	ab	ac	bc	abc	I
1	−1	−1	−1	−1	−1	−1	1	1	1	−1	1
2	−1	−1	1	−1	−1	1	1	−1	−1	1	1
3	−1	1	−1	−1	1	−1	−1	1	−1	1	1
4	−1	1	1	−1	1	1	−1	−1	1	−1	1
5	1	−1	−1	1	−1	−1	−1	−1	1	1	1
6	1	−1	1	1	−1	1	−1	1	−1	−1	1
7	1	1	−1	1	1	−1	1	−1	−1	−1	1
8	1	1	1	1	1	1	1	1	1	1	1

a full factorial 2^2

	A	B	C	a	b	c	ab	ac	bc	abc	I
2	−1	−1	1	−1	−1	1	1	−1	−1	1	1
3	−1	1	−1	−1	1	−1	−1	1	−1	1	1
5	1	−1	−1	1	−1	−1	−1	−1	1	1	1
8	1	1	1	1	1	1	1	1	1	1	1

design matrix of the 2^{3-1}, $I = abc$

model matrix — unidetifiable / identifiable

model matrix formed by all effects of A & B

→ the analyses for full factorial design can be applied on fractional factorial

- effect^2=I

  a^2=b^2=c^2=(ab)^2=(ac)^2

  =(bc)^2=(abc)^2=I

- defining relation

  I = abc (defining word)

- effect aliasing

  $a = a^2bc = bc$

  a=bc
  b=ac
  c=ab

  an alias set

- un-aliased effects are orthogonal

$E(y) = \beta_0 + \beta_a x_a + \beta_b x_b + \beta_c x_c + \beta_{ab} x_{ab} + \beta_{bc} x_{bc} + \beta_{ac} x_{ac} + \beta_{abc} x_{abc}$

$\rightarrow (\beta_0 + \beta_{abc}) + (\beta_a + \beta_{bc}) x_a + (\beta_b + \beta_{ac}) x_b + (\beta_c + \beta_{ab}) x_{ab}$