Leaf Spring Experiment

- Five factors at two levels, use a 16-run design with three replicates for each run. It is a 2^{5-1} design, 1/2 fraction of the 2^5 design.

Table 1: Factors and Levels, Leaf Spring Experiment

Factor		Level −	Level +
B.	high heat temperature (°F)	1840	1880
C.	heating time (seconds)	23	25
D.	transfer time (seconds)	10	12
E.	hold down time (seconds)	2	3
Q.	quench oil temperature (°F)	130-150	150-170

- response y = free height of spring, target = 8.0 inches.

Goal : get y as close to 8.0 as possible (nominal-the-best problem).

Leaf Spring Experiment: Design Matrix and Data

Table 2: Design Matrix and Free Height Data, Leaf Spring Experiment

B	C	D	E	Q	Free Height (y)			$\bar{y}_i$	s_i^2	$\ln s_i^2$
−	+	+	−	−	7.78	7.78	7.81	7.7900	0.0003	−8.1117
+	+	+	+	−	8.15	8.18	7.88	8.0700	0.0273	−3.6009
−	−	+	+	−	7.50	7.56	7.50	7.5200	0.0012	−6.7254
+	−	+	−	−	7.59	7.56	7.75	7.6333	0.0104	−4.5627
−	+	−	+	−	7.94	8.00	7.88	7.9400	0.0036	−5.6268
+	+	−	−	−	7.69	8.09	8.06	7.9467	0.0496	−3.0031
−	−	−	−	−	7.56	7.62	7.44	7.5400	0.0084	−4.7795
+	−	−	+	−	7.56	7.81	7.69	7.6867	0.0156	−4.1583
−	+	+	−	+	7.50	7.25	7.12	7.2900	0.0373	−3.2888
+	+	+	+	+	7.88	7.88	7.44	7.7333	0.0645	−2.7406
−	−	+	+	+	7.50	7.56	7.50	7.5200	0.0012	−6.7254
+	−	+	−	+	7.63	7.75	7.56	7.6467	0.0092	−4.6849
−	+	−	+	+	7.32	7.44	7.44	7.4000	0.0048	−5.3391
+	+	−	−	+	7.56	7.69	7.62	7.6233	0.0042	−5.4648
−	−	−	−	+	7.18	7.18	7.25	7.2033	0.0016	−6.4171
+	−	−	+	+	7.81	7.50	7.59	7.6333	0.0254	−3.6717

Why Use Fractional Factorial Designs (FFDs)?

- If a 2^5 design is used for the experiment, its 31 degrees of freedom would be allocated as follows:

	Main Effects	Interactions 2-Factor	3-Factor	4-Factor	5-Factor
#	5	10	10	5	1

- Using effect hierarchy principle, one would argue that 4fi's, 5fi and even 3fi's are not likely to be important. There are 10+5+1 = 16 (16/32 = 1/2) such effects, half of the total runs! Using a 2^5 design can be wasteful (unless 32 runs cost about the same as 16 runs.)
- Use of an FFD instead of full factorial design is usually done for economic reasons. Since there is **no free lunch** , what **price to pay**? See next slide.

❖ **Reading**: textbook, 5.1

Effect Aliasing and Defining Relation

- In the design matrix, col $B\times$ col $C\times$ col D = col E. That means,

$$\bar{y}(E+)-\bar{y}(E-)=\bar{y}(BCD+)-\bar{y}(BCD-).$$

Therefore the design is not capable of distinguishing E from BCD. The main effect E is **aliased** with the interaction BCD. Notationally,

$$E=BCD \quad \text{or} \quad \mathbf{I}=BCDE,$$

where **I** (= column of $+$'s) is the identity element in the group of multiplications.
(Notice the mathematical similarity between aliasing and confounding. What is the difference?)

- $\mathbf{I}=BCDE$ is the **defining relation** for the 2^{5-1} design.
It implies all the 15 effect aliasing relations :

$$B=CDE,\ C=BDE,\ D=BCE,\ E=BCD,$$
$$BC=DE,\ BD=CE,\ BE=CD,$$
$$Q=BCDEQ,\ BQ=CDEQ,\ CQ=BDEQ,\ DQ=BCEQ,$$
$$EQ=BCDQ,\ BCQ=DEQ,\ BDQ=CEQ,\ BEQ=CDQ.$$

	A	B	C	a	b	c	ab	ac	bc	abc
1	−1	−1	−1	−1	−1	−1	1	1	1	−1
2	−1	−1	1	−1	−1	1	1	−1	−1	1
3	−1	1	−1	−1	1	−1	−1	1	−1	1
4	−1	1	1	−1	1	1	−1	−1	1	−1
5	1	−1	−1	1	−1	−1	−1	−1	1	1
6	1	−1	1	1	−1	1	−1	1	−1	−1
7	1	1	−1	1	1	−1	1	−1	−1	−1
8	1	1	1	1	1	1	1	1	1	1

	A	B	C	a	b	c	ab	ac	bc	abc
2	−1	−1	1	−1	−1	1	1	−1	−1	1
3	−1	1	−1	−1	1	−1	−1	1	−1	1
5	1	−1	−1	1	−1	−1	−1	−1	1	1
8	1	1	1	1	1	1	1	1	1	1

- $\text{effect}^2 = I$

  $a^2 = b^2 = c^2 = (ab)^2 = (ac)^2$

  $= (bc)^2 = (abc)^2 = I$

- defining relation

  $I = abc$

- effect aliasing

  $a = bc$

  $b = ac$

  $c = ab$

- un-aliased effects are orthogonal

	A	B	C	D	a	b	c	d	ab	ac	ad	bc	bd	cd	abc	abd	acd	bcd	abcd
1	−1	−1	−1	−1	−1	−1	−1	−1	1	1	1	1	1	1	−1	−1	−1	−1	1
2	−1	−1	−1	1	−1	−1	−1	1	1	1	−1	1	−1	−1	−1	1	1	1	−1
3	−1	−1	1	−1	−1	−1	1	−1	1	−1	1	−1	1	−1	1	−1	1	1	−1
4	−1	−1	1	1	−1	−1	1	1	1	−1	−1	−1	−1	1	1	1	−1	−1	1
5	−1	1	−1	−1	−1	1	−1	−1	−1	1	1	−1	−1	1	1	1	−1	1	−1
6	−1	1	−1	1	−1	1	−1	1	−1	1	−1	−1	1	−1	1	−1	1	−1	1
7	−1	1	1	−1	−1	1	1	−1	−1	−1	1	1	−1	−1	−1	1	1	−1	1
8	−1	1	1	1	−1	1	1	1	−1	−1	−1	1	1	1	−1	−1	−1	1	−1
9	1	−1	−1	−1	1	−1	−1	−1	−1	−1	−1	1	1	1	1	1	1	−1	−1
10	1	−1	−1	1	1	−1	−1	1	−1	−1	1	1	−1	−1	1	−1	−1	1	1
11	1	−1	1	−1	1	−1	1	−1	−1	1	−1	−1	1	−1	−1	1	−1	1	1
12	1	−1	1	1	1	−1	1	1	−1	1	1	−1	−1	1	−1	−1	1	−1	−1
13	1	1	−1	−1	1	1	−1	−1	1	−1	−1	−1	−1	1	−1	−1	1	1	1
14	1	1	−1	1	1	1	−1	1	1	−1	1	−1	1	−1	−1	1	−1	−1	−1
15	1	1	1	−1	1	1	1	−1	1	1	−1	1	−1	−1	1	−1	−1	−1	−1
16	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

Clear Effects

- A main effect or two-factor interaction (2fi) is called **clear** if it is not aliased with any other m.e.'s or 2fi's and **strongly clear** if it is not aliased with any other m.e.'s, 2fi's or 3fi's. Therefore a clear effect is *estimable* under the assumption of negligible 3-factor and higher-order interactions and a strongly clear effect is *estimable* under the weaker assumption of negligible 4-factor and higher-order interactions.

- In the 2^{5-1} design with $\mathbf{I} = BCDE$, which effects are clear and strongly clear?

  Ans: B, C, D, E are clear, Q, BQ, CQ, DQ, EQ are strongly clear.

- Consider the alternative plan 2^{5-1} design with $\mathbf{I} = BCDEQ$. (It is said to have resolution V because the length of the defining word is 5 while the previous plan has resolution IV.) It can be verified that all five main effects are strongly clear and all 10 2fi's are clear. (*Do the derivations*). This is a very good plan because *each* of the 15 degrees of freedom is either clear or strongly clear.

Defining Contrast Subgroup for 2^{k-p} Designs

- A 2^{k-p} design has k factors, 2^{k-p} runs, and it is a 2^{-p}th fraction of the 2^k design. The fraction is defined by p *independent* defining words. The group formed by these p words is called the **defining contrast subgroup**. It has $2^p - 1$ words plus the identity element **I**.

- **Resolution** = shortest wordlength among the $2^p - 1$ words.

- Example: A 2^{6-2} design with $\mathbf{5} = \mathbf{12}$ and $\mathbf{6} = \mathbf{134}$. The two independent defining words are $\mathbf{I} = \mathbf{125}$ and $\mathbf{I} = \mathbf{1346}$. Then $\mathbf{I} = \mathbf{125} \times \mathbf{1346} = \mathbf{23456}$. The defining contrast subgroup = $\{\mathbf{I}, \mathbf{125}, \mathbf{1346}, \mathbf{23456}\}$. The design has resolution III.

Deriving Aliasing Relations for the 2^{6-2} design

- For the same 2^{6-2} design, the defining contrast subgroup is

$$\mathbf{I} = \mathbf{125} = \mathbf{1346} = \mathbf{23456}.$$

I	=	**125**	=	**1346**	=	**23456**,	
1	=	**25**	=	**346**	=	**123456**,	
2	=	**15**	=	**12346**	=	**3456**,	
3	=	**1235**	=	**146**	=	**2456**,	
4	=	**1245**	=	**136**	=	**2356**,	
5	=	**12**	=	**13456**	=	**2346**,	
6	=	**1256**	=	**134**	=	**2345**,	
13	=	**235**	=	**46**	=	**12456**,	
14	=	**245**	=	**36**	=	**12356**,	(1)
16	=	**256**	=	**34**	=	**12345**,	
23	=	**135**	=	**1246**	=	**456**,	
24	=	**145**	=	**1236**	=	**356**,	
26	=	**156**	=	**1234**	=	**345**,	
35	=	**123**	=	**1456**	=	**246**,	
45	=	**124**	=	**1356**	=	**236**,	
56	=	**126**	=	**1345**	=	**234**.	

All the 15 degrees of freedom (each is a coset in group theory) are identified.

- It has the clear effects: **3, 4, 6, 23, 24, 26, 35, 45, 56**. It has resolution III.

WordLength Pattern and Resolution

- Define A_i = number of defining words of length i.
- $W = (A_3, A_4, A_5, \ldots)$ is called the **wordlength pattern**.
  - In this design, W = (1, 1, 1, 0).
  - It is required that $A_1 = 0$ and $A_2 = 0$. (Why? No main effect is allowed to be aliased with the intercept or another main effect.)
- **Resolution** = smallest r such that $A_r \geq 1$.
- **Maximum resolution criterion**: For fixed k and p, choose a 2^{k-p} design with maximum resolution.
- **Rules for Resolution IV and V Designs:**

(i) In any resolution IV design, the main effects are clear.

(ii) In any resolution V design, the main effects are strongly clear and the two-factor interactions are clear. (2)

(iii) Among the resolution IV designs with given k and p, those with the largest number of clear two-factor interactions are the best.

A Projective Rationale for Resolution

- For a resolution R design, its projection onto any $R-1$ factors is a full factorial in the $R-1$ factors. This would allow *effects of all orders among the $R-1$ factors to be estimable*. (**Caveat**: it makes the assumption that other factors are inert.)

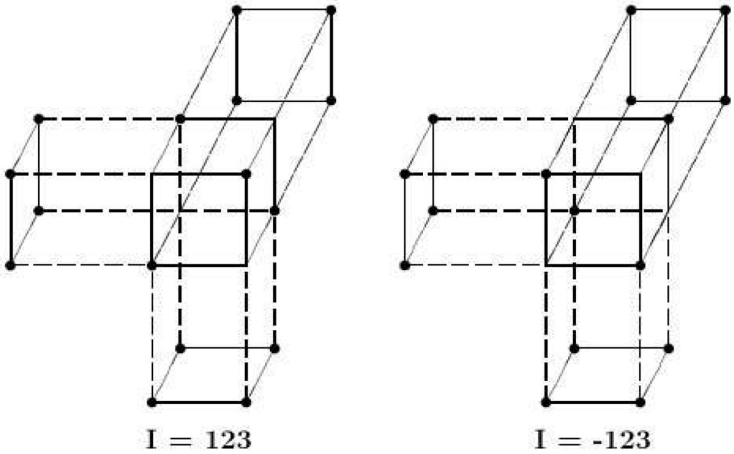

Figure 1: 2^{3-1} Designs Using $\mathbf{I} = \pm\mathbf{123}$ and Their Projections to 2^2 Designs.

Minimum Aberration Criterion

- Motivating example: consider the two 2^{7-2} designs:

$$d_1: \quad \mathbf{I} = \mathbf{4567} = \mathbf{12346} = \mathbf{12357},$$
$$d_2: \quad \mathbf{I} = \mathbf{1236} = \mathbf{1457} = \mathbf{234567}.$$

  - Both have resolution IV.
  - But $W(d_1) = (0,1,2,0,0)$ and $W(d_2) = (0,2,0,1,0)$.
  - Which one is better?
  - Intuitively one would argue that d_1 is better because
    $A_4(d_1) = 1 < A_4(d_2) = 2$.
    (Why? Effect hierarchy principle.)
- For any two 2^{k-p} designs d_1 and d_2, let
  r = the smallest integer such that $A_r(d_1) \neq A_r(d_2)$.
  - d_1 is said to have *less aberration* than d_2 if $A_r(d_1) < A_r(d_2)$.
  - If no design has less aberration than d_1, then d_1 has *minimum aberration*.
- Throughout the book, this is the *major criterion used for selecting fractional factorial designs*. Its theory is covered in the Mukherjee-Wu (2006) book.

❖ **Reading**: textbook, 5.2

Analysis for Location Effects

- *Same* strategy as in full factorial experiments *except* for the **interpretation and handling of aliased effects**.
- For the location effects (based on $\bar{y}_i$ values),
  - the estimated factorial effects are given in Table 3 (LNp.6-14), and
  - the corresponding half-normal plot in Figure 2 (LNp.6-15).
- Visually one may judge that
  - Q, B, C, CQ and possibly E, BQ are significant.

  One can apply the studentized maximum modulus test (see textbook, sec. 4.14, not covered in class) to confirm that Q, B, C, CQ are significant at 0.05 level (see textbook, p.219 and 221).
- The $B \times Q$ and $C \times Q$ plots (Figure 3, LNp.6-16) show that they are synergystic.
- For illustration, we use the model

$$\hat{y} = 7.6360 + 0.1106x_B + 0.0519x_E + 0.0881x_C - 0.1298x_Q + 0.0423x_Bx_Q - 0.0827x_Cx_Q \quad (3)$$

Factorial Effects, Leaf Spring Experiment

Table 3: Factorial Effects, Leaf Spring Experiment

Effect	$\bar{y}$	$\ln s^2$
B	0.221	1.891
C	0.176	0.569
D	0.029	−0.247
E	0.104	0.216
Q	−0.260	0.280
BQ	0.085	−0.589
CQ	−0.165	0.598
DQ	0.054	1.111
EQ	0.027	0.129
BC	0.017	−0.002
BD	0.020	0.425
CD	−0.035	0.670
BCQ	0.010	−1.089
BDQ	−0.040	−0.432
BEQ	−0.047	0.854

Half-normal Plot of Location Effects, Leaf Spring Experiment

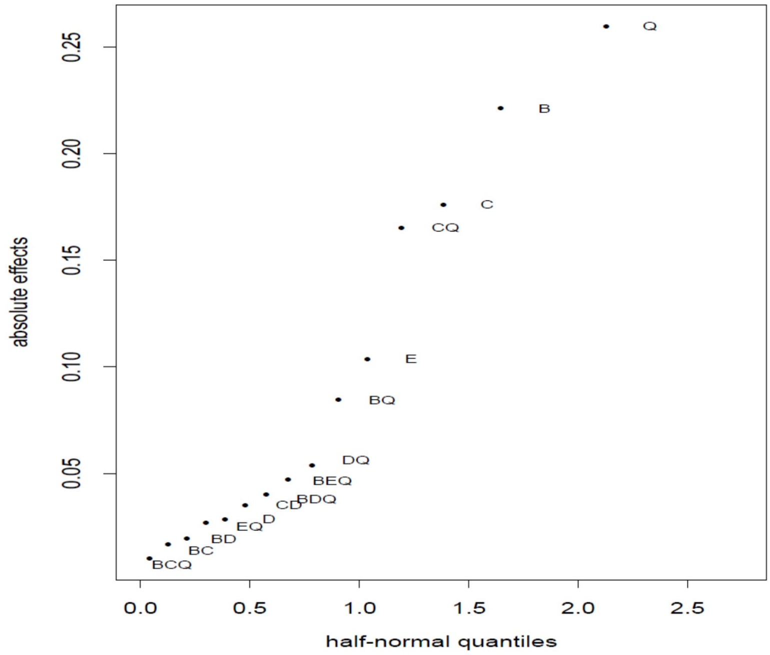

Figure 2: Half-Normal Plot of Location Effects, Leaf Spring Experiment

Interaction Plots

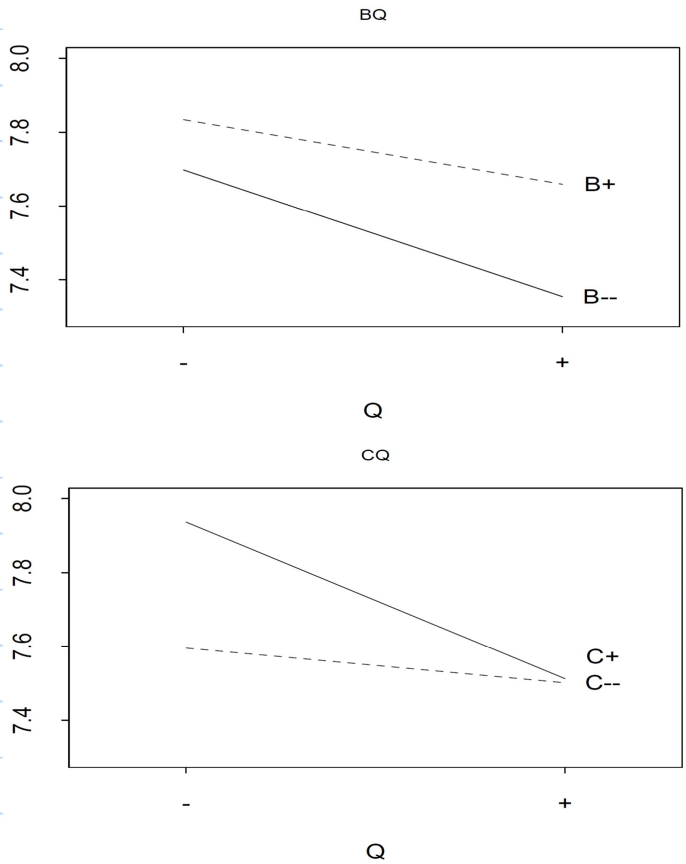

Figure 3: $B \times Q$ and $C \times Q$ interaction plots, Leaf Spring Experiment

Analysis for Dispersion Effects

- For the dispersion effects (based on $z_i = \ln s_i^2$ values),
  - the estimated factorial effects are given in Table 3 (LNp.6-14)
  - the half-normal plot is given in Figure 4 (LNp.6-18).
- Visually only effect B stands out. This is confirmed by applying the studentized maximum modulus test (see textbook, sec. 4.14).
- For illustration, we will include B, DQ, BCQ in the following model,

$$\ln \hat{\sigma}^2 = -4.9313 + 0.9455 x_B + 0.5556 x_D x_Q - 0.5445 x_B x_C x_Q. \qquad (4)$$

- The $D \times Q$ and $B \times C \times Q$ interaction plots are given in Figures 5 and 6 (LNp.6-19).

Half-normal Plot of Dispersion Effects, Leaf Spring Experiment

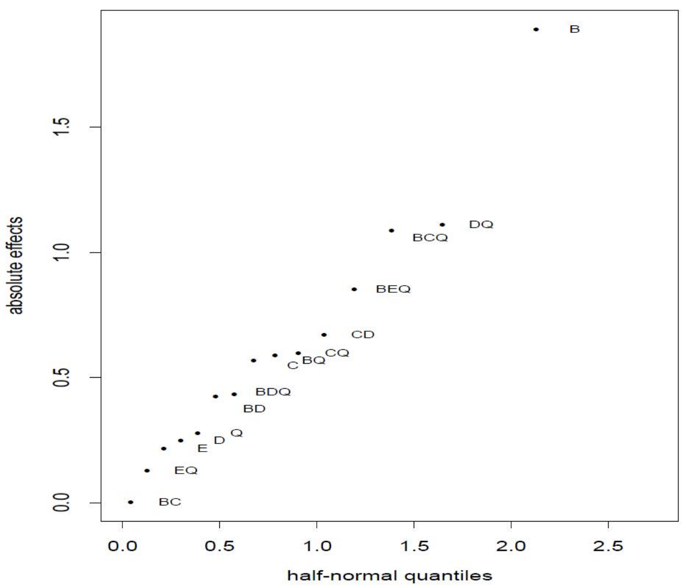

Figure 4: Half-Normal Plot of Dispersion Effects, Leaf Spring Experiment

Interaction Plots for Dispersion Effects

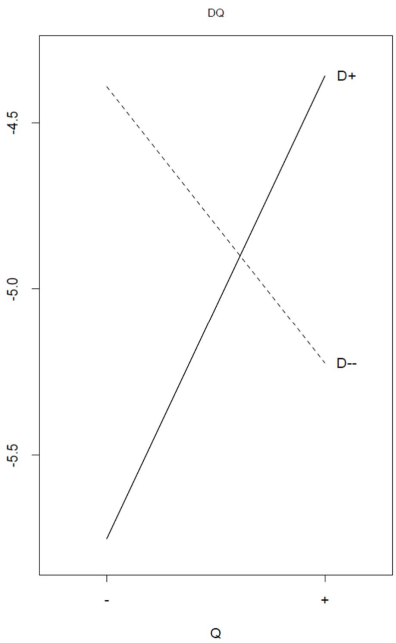

Figure 5 : $D \times Q$ Interaction Plot

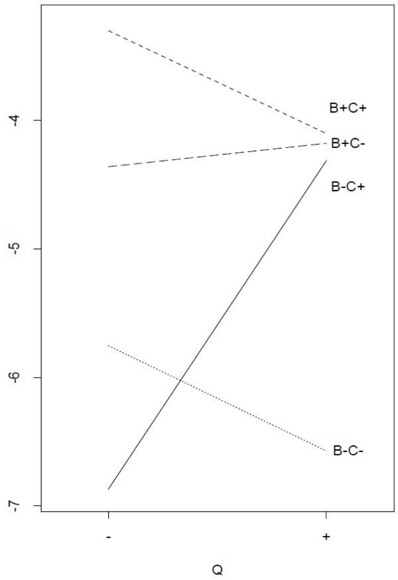

Figure 6 : $B \times C \times Q$ Interaction Plot

Two-Step Procedure for Optimization

- Step 1: To minimize s^2 (or $\ln s^2$), we can
  - choose $B = -$ based on eq. (4) in LNp.6-17,
  - choose the combination with the lowest value, $D = +$, $Q = -$ based on the $D \times Q$ plot (Figure 5, LNp.6-19),
  - with $B = -$ and $Q = -$, choose $C = +$ to attain the minimum in the $B \times C \times Q$ interaction plot (Figure 6, LNp.6-19).

  Another confirmation: they lead to $x_B = -, x_D x_Q = -$ and $x_B x_C x_Q = +$ in the model (4), which make each of the last three terms negative.

- Step 2: With $(B, C, D, Q) = (-, +, +, -)$, from model (3) in LNp.6-13, we have

$$\begin{aligned} \hat{y} &= 7.6360 + 0.1106(-1) + 0.0519 x_E + 0.0881(+1) - 0.1298(-1) \\ &\quad + 0.0423(-1)(-1) - 0.0827(+1)(-1) \\ &= 7.8683 + 0.0519 x_E. \end{aligned}$$

  By solving $\hat{y} = 8.0$, $x_E = 2.54$.

  Warning: This is way outside the experimental range for factor E. Such a value may not make physical sense and the predicted variance value for this setting may be too optimistic and not substantiated.

❖ **Reading**: textbook, 5.3

Techniques for Resolving Ambiguities in Aliased Effects

- Among the three factorial effects that feature in model (4) (LNp.6-17), B is clear and DQ is strongly clear.

- However, the term $x_B x_C x_Q$ is aliased with $x_D x_E x_Q$ (See bottom of LNp. 6-4). The following three techniques can be used to resolve the ambiguities.
  - *Subject matter knowledge* may suggest some effects in the alias set are not likely to be significant (or does not have a good physical interpretation).
  - Or use *effect hierarchy principle* to *assume away* some higher order effects.
  - Or use a **follow-up experiment** to **de-alias** these effects. Two methods are given in section 5.4 of textbook.

Fold-over Technique

- Suppose the original experiment is based on a 2^{7-4}_{III} design with generators

$$d_1: \ \mathbf{4} = \mathbf{12}, \ \mathbf{5} = \mathbf{13}, \ \mathbf{6} = \mathbf{23}, \ \mathbf{7} = \mathbf{123}.$$

None of its main effects are clear.

- To de-alias them, we can choose another 8 runs (no. 9-16 in Table 4, LNp.6-23) with **reversed** signs for each of the 7 factors. This follow-up design d_2 has the generators

$$d_2: \mathbf{4}' = -\mathbf{1}'\mathbf{2}', \mathbf{5}' = -\mathbf{1}'\mathbf{3}', \mathbf{6}' = -\mathbf{2}'\mathbf{3}', \mathbf{7}' = \mathbf{1}'\mathbf{2}'\mathbf{3}'.$$

With the extra degrees of freedom, we can introduce a new factor **8** (or a blocking variable) for run number 1-8, and $-\mathbf{8}$ for run number 9-16. See Table 4.

- The combined design $d_1 + d_2$ is a 2^{8-4}_{IV} design and thus all main effects are clear. (Its defining contrast subgroup is on textbook, p.227).

Augmented Design Matrix Using Fold-over Technique

Table 4: Augmented Design Matrix Using Fold-Over Technique

				d_1				
Run	**1**	**2**	**3**	**4=12**	**5=13**	**6=23**	**7=123**	**8**
1	−	−	−	+	+	+	−	+
2	−	−	+	+	−	−	+	+
3	−	+	−	−	+	−	+	+
4	−	+	+	−	−	+	−	+
5	+	−	−	−	−	+	+	+
6	+	−	+	−	+	−	−	+
7	+	+	−	+	−	−	−	+
8	+	+	+	+	+	+	+	+
				d_2				
Run	**−1**	**−2**	**−3**	**−4**	**−5**	**−6**	**−7**	**−8**
9	+	+	+	−	−	−	+	−
10	+	+	−	−	+	+	−	−
11	+	−	+	+	−	+	−	−
12	+	−	−	+	+	−	+	−
13	−	+	+	+	+	−	−	−
14	−	+	−	+	−	+	+	−
15	−	−	+	−	+	+	+	−
16	−	−	−	−	−	−	−	−

Fold-over Technique: Version Two

- Suppose one factor, say **5**, is very important.
  We want to de-alias **5** and all 2fi's involving **5**.
- Choose, instead, the following 2^{7-4}_{III} design

  $$d_3 : \mathbf{4}' = \mathbf{1}'\mathbf{2}', \mathbf{5}' = -\mathbf{1}'\mathbf{3}', \mathbf{6}' = \mathbf{2}'\mathbf{3}', \mathbf{7}' = \mathbf{1}'\mathbf{2}'\mathbf{3}'.$$

  Then the combined design $d_1 + d_3$ is a 2^{7-3}_{III} design with the generators

  $$d' : \mathbf{4} = \mathbf{12}, \mathbf{6} = \mathbf{23}, \mathbf{7} = \mathbf{123}. \qquad (5)$$

  Since **5** does not appear in (5), **5** is strongly clear and all 2fi's involving **5** are clear. However, other main effects are not clear (see Table 5.7 in textbook, p.228, for $d_1 + d_3$).
- Choice between d_2 and d_3 depends on the priority given to the effects.

Critique of Fold-over Technique

- Fold-over technique is not an efficient technique.
  - It requires doubling of the run size and can only de-alias a *specific* set of effects.
  - In practice, after analyzing the first experiment, a set of effects will emerge and need to be de-aliased.
  - It will usually require much *fewer* runs to de-alias a few effects.

- A more efficient technique that does not have these deficiencies is the optimum design approach given in Section 5.4.2.

❖ **Reading**: textbook, 5.4.1

Use of Design Tables

- Minimum aberration (MA) designs are given in the tables in textbook, Appendix 5A. If two designs are given for same k and p,
  - the first is an MA design and
  - the second is better in having a larger number of clear effects.

  Two tables are given on next two slides.
- In Table 7 (LNp.6-28),
  - the first 2^{9-4} design has MA and 8 clear 2fi's, and
  - the second 2^{9-4} design is
    * the second best according to the MA criterion,
    * but has 15 clear 2fi's.

  Using Rule (iii) in (2) on LNp.6-10, the second design is better because both have resolution IV (Details given on p. 234 of textbook).
- It is not uncommon to find a design with slightly worse aberration but more clear effects. Thus **the number of clear effects** should be used as a *supplementary criterion* to the MA criterion.

Table 6: 16-Run 2^{k-p} FFD $(k-p=4)$

(k is the number of factors and F&R is the fraction and resolution.)

k	F&R	Design Generators	Clear Effects
5	2_{V}^{5-1}	$5=1234$	all five main effects, all 10 2fi's
6	2_{IV}^{6-2}	$5=123, 6=124$	all six main effects
6^*	2_{III}^{6-2}	$5=12, 6=134$	3, 4, 6, 23, 24, 26, 35, 45, 56
7	2_{IV}^{7-3}	$5=123, 6=124, 7=134$	all seven main effects
8	2_{IV}^{8-4}	$5=123, 6=124, 7=134, 8=234$	all eight main effects
9	2_{III}^{9-5}	$5=123, 6=124, 7=134, 8=234, 9=1234$	none
10	2_{III}^{10-6}	$5=123, 6=124, 7=134, 8=234, 9=1234, t_0=34$	none
11	2_{III}^{11-7}	$5=123, 6=124, 7=134, 8=234, 9=1234, t_0=34, t_1=24$	none
12	2_{III}^{12-8}	$5=123, 6=124, 7=134, 8=234, 9=1234, t_0=34, t_1=24, t_2=14$	none
13	2_{III}^{13-9}	$5=123, 6=124, 7=134, 8=234, 9=1234, t_0=34, t_1=24, t_2=14, t_3=23$	none
14	2_{III}^{14-10}	$5=123, 6=124, 7=134, 8=234, 9=1234, t_0=34, t_1=24, t_2=14, t_3=23, t_4=13$	none
15	2_{III}^{15-11}	$5=123, 6=124, 7=134, 8=234, 9=1234, t_0=34, t_1=24, t_2=14, t_3=23, t_4=13, t_5=12$	none

Table 7: 32 Run 2^{k-p} FFD $(k-p=5, 6\le k\le 11)$

(k is the number of factors and F&R is the fraction and resolution.)

k	F&R	Design Generators	Clear Effects
6	2_{VI}^{6-1}	$6=12345$	all six main effects, all 15 2fi's
7	2_{IV}^{7-2}	$6=123, 7=1245$	all seven main effects, 14, 15, 17, 24, 25, 27, 34, 35, 37, 45, 46, 47, 56, 57, 67
8	2_{IV}^{8-3}	$6=123, 7=124, 8=1345$	all eight main effects, 15, 18, 25, 28, 35, 38, 45, 48, 56, 57, 58, 68, 78
9	2_{IV}^{9-4}	$6=123, 7=124, 8=125, 9=1345$	all nine main effects, 19, 29, 39, 49, 59, 69, 79, 89
9	2_{IV}^{9-4}	$6=123, 7=124, 8=134, 9=2345$	all nine main effects, 15, 19, 25, 29, 35, 39, 45, 49, 56, 57, 58, 59, 69, 79, 89
10	2_{IV}^{10-5}	$6=123, 7=124, 8=125, 9=1345, t_0=2345$	all 10 main effects
10	2_{III}^{10-5}	$6=12, 7=134, 8=135, 9=145, t_0=345$	3, 4, 5, 7, 8, 9, t_0, 23, 24, 25, 27, 28, 29, $2t_0$, 36, 46, 56, 67, 68, 69, $6t_0$
11	2_{IV}^{11-6}	$6=123, 7=124, 8=134, 9=125, t_0=135, t_1=145$	all 11 main effects
11	2_{III}^{11-6}	$6=12, 7=13, 8=234, 9=235, t_0=245, t_1=1345$	4, 5, 8, 9, t_0, t_1, 14, 15, 18, 19, $1t_0$, $1t_1$

Choice of Fractions and Avoidance of Specific Level Combinations

- A 2^{k-p} design has 2^p choices.
- In general, use randomization to choose one of them.
  For example, the 2^{7-3} design has 8 choices

$$4 = \pm 12,\ 5 = \pm 13,\ 6 = \pm 23.$$

  Randomly choose the signs.
- If specific level combinations, e.g.,

$$(+,+,+) \text{ for high pressure, high temperature, high concentration,}$$

  are deemed undesirable or even disastrous, they can be avoided by choosing a fraction that does not contain them. Example on p.237 of textbook.

❖ **Reading**: textbook, 5.5

Blocking in FF Designs

- Example: Arrange the 2^{6-2} design in four ($= 2^2$) blocks with

$$\mathbf{I} = \mathbf{1235} = \mathbf{1246} = \mathbf{3456}.$$

- Suppose we choose

$$b_1 = \mathbf{134},\quad b_2 = \mathbf{234},\quad b_1 b_2 = \mathbf{12}.$$

- Then

$$\begin{aligned} b_1 &= \mathbf{134} = \mathbf{245} = \mathbf{236} = \mathbf{156}, \\ b_2 &= \mathbf{234} = \mathbf{145} = \mathbf{136} = \mathbf{256}, \\ b_1 b_2 &= \mathbf{12} = \mathbf{35} = \mathbf{46} = \mathbf{123456}; \end{aligned}$$

$$\begin{array}{ccccccc} \mathbf{13} & = & \mathbf{25} & = & \mathbf{2346} & = & \mathbf{1456}, \\ \mathbf{14} & = & \mathbf{26} & = & \mathbf{2345} & = & \mathbf{1356}, \\ \mathbf{15} & = & \mathbf{23} & = & \mathbf{2456} & = & \mathbf{1346}, \\ \mathbf{16} & = & \mathbf{24} & = & \mathbf{2356} & = & \mathbf{1345}, \\ \mathbf{34} & = & \mathbf{56} & = & \mathbf{1245} & = & \mathbf{1236}, \\ \mathbf{36} & = & \mathbf{45} & = & \mathbf{1256} & = & \mathbf{1234}. \end{array}$$

The $4 \times 3 = 12$ factorial effects are confounded with block effects and cannot be used for estimation. Among the remaining 12 degrees of freedom, six are main effects and the rest are given above.

Use of Design Tables for Blocking

- Among the 15 degrees of freedom for the blocked design on LNp.6-30, 3 are allocated for block effects and 6 are for clear main effects (see Table 8 in LNp.6-32). The remaining 6 degrees of freedom are six pairs of aliased two-factor interactions.
- For the 2^{6-2} design with $5 = 12$, $6 = 134$, if we use the block generators $b_1 = 13$, $b_2 = 14$, there are a total of 9 clear effects (see Table 8 in LNp.6-32):

$$3, 4, 6, 23, 24, 26, 35, 45, 56.$$

  - Thus, the total number of clear effects for this blocked design is 3 more than the total number of clear effects for the blocked design on LNp.6-30.
  - However, only the main effects 3, 4, 6 are clear.

Table 8: Sixteen-Run 2^{k-p} Fractional Factorial Designs in 2^q Blocks

k	p	q	Design Generators	Block Generators	Clear Effects
5	1	1	$5 = 1234$	$b_1 = 12$	all five main effects, all 2fi's except 12
5	1	2	$5 = 1234$	$b_1 = 12$, $b_2 = 13$	all five main effects, 14, 15, 24, 25, 34, 35, 45
5	1	3	$5 = 123$	$b_1 = 14$, $b_2 = 24$, $b_3 = 34$	all five main effects
6	2	1	$5 = 123, 6 = 124$	$b_1 = 134$	all six main effects
6	2	1	$5 = 12, 6 = 134$	$b_1 = 13$	3, 4, 6, 23, 24, 26, 35, 45, 56
6	2	2	$5 = 123, 6 = 124$	$b_1 = 134$, $b_2 = 234$	all six main effects
6	2	2	$5 = 12, 6 = 134$	$b_1 = 13$, $b_2 = 14$	3, 4, 6, 23, 24, 26, 35, 45, 56
6	2	3	$5 = 123, 6 = 124$	$b_1 = 13$, $b_2 = 23$, $b_3 = 14$	all six main effects

Table 8: Sixteen-Run 2^{k-p} Fractional Factorial Designs in 2^q Blocks (Cont.)

k	p	q	Design Generators	Block Generators	Clear Effects
7	3	1	$5 = 123, 6 = 124$, $7 = 134$	$b_1 = 234$	all seven main effects
7	3	2	$5 = 123, 6 = 124$, $7 = 134$	$b_1 = 12$, $b_2 = 13$	all seven main effects
7	3	3	$5 = 123, 6 = 124$, $7 = 134$	$b_1 = 12$, $b_2 = 13$, $b_3 = 14$	all seven main effects
8	4	1	$5 = 123, 6 = 124$, $7 = 134, 8 = 234$	$b_1 = 12$	all eight main effects
8	4	2	$5 = 123, 6 = 124$, $7 = 134, 8 = 234$	$b_1 = 12$, $b_2 = 13$	all eight main effects
8	4	3	$5 = 123, 6 = 124$, $7 = 134, 8 = 234$	$b_1 = 12$, $b_2 = 13$, $b_3 = 14$	all eight main effects
9	5	1	$5 = 12, 6 = 13$, $7 = 14$, $8 = 234$, $9 = 1234$	$b_1 = 23$	none
9	5	2	$5 = 12, 6 = 13$, $7 = 14, 8 = 234$, $9 = 1234$	$b_1 = 23$, $b_2 = 24$	none

- More FF designs in blocks are given in Appendix 5B of textbook.

❖ **Reading**: textbook, 5.6