2^k Designs in 2^q Blocks

one block factor, 2^q levels
block size 2^{k-q} ($\because$ total 2^k EUs)

k treatment factors. each 2 levels
full factorial $\Rightarrow$ 2^k treatments ($\Rightarrow$ need 2^k EUs)

block size 2^{k-q}
$< 2^k$ treatments
$\Rightarrow$ incomplete blocking

- Example: Arranging a 2^3 design in 2 blocks (of size 4). $\frac{2^3}{2}$ Use the 123 column in Table 9 (LNp.5-37) to define the blocking scheme:

  block I if 123 = −, and block II if 123 = +.

★ conceptual model:
$Z \sim \beta_0 +$ block effects ($2^q - 1$ parameters)
$+$ all treatment effects $+ \varepsilon$ ($2^k - 1$ parameters)
$\Rightarrow$ no interaction btwn block & treatment factors

- The block effect estimate $\bar{y}(II) - \bar{y}(I)$ is identical to the estimate of the 123 interaction $\bar{y}(123 = +) - \bar{y}(123 = -)$. The block effect b and the interaction 123 are called **confounded**. Notationally,

  cf. aliased (Ch5)

  $$b = \mathbf{123}.$$

  ① In design matrix, column b = (column 1)×(column 2)×(column 3)
  ② In model matrix, block effect column = 123 interaction column
  ③ $\beta_b + \beta_{123}$ is jointly estimated

- By giving up (sacrifying) the ability to estimate 123, this blocking scheme increases the precision in the estimates of main effects and 2fi's by arranging 8 runs in two *homogeneous* blocks.

  $\because$ the block effect ① can reduce $\hat{\sigma}^2$ ② is orthogonal to other treatment effects (their estimates not biased by block effect)

  a good choice?

- Why sacrificing 123?

ans: Effect hierarchy principle. → 123 is the least important effect among all the 8 factorial effects.

Arrangement of 2^3 Design in 2 Blocks

block 1: EU_{11} EU_{13} EU_{12} EU_{14} — (1,2,3)
block 2: EU_{21} EU_{23} EU_{22} EU_{24} — (1,2,3)

④ ← how? → ④

8 treatments
(−,−,−)
(−,−,+)
⋮
(+,+,+)

conceptual model: (Note. not interested in β_b)

$$Z = \beta_0 + \beta_b x_b + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_{12} x_{12} + \beta_{13} x_{13} + \beta_{23} x_{23} + \beta_{123} x_{123} + \varepsilon$$

block effect

$$= \beta_0 + (\beta_b + \beta_{123}) x_b + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_{12} x_{12} + \beta_{13} x_{13} + \beta_{23} x_{23} + \varepsilon$$

Table 9: Arranging a 2^3 Design in Two Blocks of Size Four
(The 3 factors are denoted by **1**, **2**, and **3**)

intercept

123 ⇒ $x_b = x_{123}$

Run	I	1	2	3	12	13	23	123	Block
1	+	−	−	−	+	+	+	−	− I
2	+	−	−	+	+	−	−	+	+ II
3	+	−	+	−	−	+	−	+	+ II
4	+	−	+	+	−	−	+	−	− I
5	+	+	−	−	−	−	+	+	+ II
6	+	+	−	+	−	+	−	−	− I
7	+	+	+	−	+	−	−	−	− I
8	+	+	+	+	+	+	+	+	+ II

4 "+"
4 "−"

design matrix of block factor

# of possible choices: $\binom{8}{4} = \frac{8!}{4!4!}$

Why might not be good choices? Ans. It would be better to have orthogonality.

# of possible columns: 7 ⇒ guarantee orthogonality

model matrix (all treatment effects)

design matrix of treatment factors

combine → whole design matrix

In the model matrix, the 8 d.f. are used to estimate
β_0 (intercept),
$\beta_1, \beta_2, \beta_3$,
$\beta_{12}, \beta_{13}, \beta_{23}$,
$\beta_b + \beta_{123}$ ← a joint effect

A 2^3 Design in 4 Blocks

one block factor, 4 levels
block size 2 = 2^2
8 treatments
each with 2 levels

- Similarly we can use $b_1 = \mathbf{12}$ and $b_2 = \mathbf{13}$ to define two independent blocking variables. The 4 blocks I, II, III and IV are defined by $b_1 = \pm$ and $b_2 = \pm$:

block1: EU11, EU12; block2: EU21, EU22; block3: EU31, EU32; block4: EU41, EU42

(1,2,3) (1,2,3) (1,2,3) (1,2,3)

② ② ② ②

(−,−,−)
(−,−,+)
⋮
(+,+,+)

pseudo block factors

		b_1	
	b_2	$-$	$+$
$-$		I	III
$+$		II	IV

β_{b_1} β_{b_2} $\beta_{b_1b_2}$

block	b_1	b_2	b_1b_2
I	−	−	+
II	−	+	−
III	+	−	−
IV	+	+	+

a block factor with 4 levels needs 3 df. for its block effects.

This is not an interaction effect. It represents a block main effect.

A group structure in Algebra, with 2 generators b_1, b_2

Q: Why not sacrify 123?

$\{I, b_1, b_2, b_1b_2\}$

I	b_1	b_2	b_1b_2
123	1 ×		23
	2 ×		13
	3 ×		12
	12		3 ×
	13		2 ×
	23		1 ×

identity element → Intercept

- A 2^3 design in 4 blocks is given in Table 10 (LNp.5-39). Confounding relationships:
$b_1 = \mathbf{12}$, $b_2 = \mathbf{13}$, $b_1b_2 = \mathbf{12} \times \mathbf{13} = \mathbf{23}$. $= 1^2 23$
Thus **12**, **13** and **23** are confounded with block effects and thus sacrificed.

Note. In model matrix, for every columns x_i's, we have $x_i^2 = I$

Arranging a 2^3 Design in 4 Blocks

Table 10: Arranging a 2^3 Design in Four Blocks of Size Two

$\beta_{b_2} + \beta_{13}$ $\beta_{b_1} + \beta_{12}$ $\beta_{b_1b_2} + \beta_{23}$

$\binom{7}{2}$ choices

# of possible choices: $\binom{8}{2\,2\,2\,2}$

					b_1	b_2	b_1b_2		
Run	I	**1**	**2**	**3**	**12**	**13**	**23**	**123**	block
1	+	−	−	−	+	+	+	−	IV
2	+	−	−	+	+	−	−	+	III
3	+	−	+	−	−	+	−	+	II
4	+	−	+	+	−	−	+	−	I
5	+	+	−	−	−	−	+	+	I
6	+	+	−	+	−	+	−	−	II
7	+	+	+	−	+	−	−	−	III
8	+	+	+	+	+	+	+	+	IV

2 "I" 2 "II" 2 "III" 2 "IV"

model matrix (treatment effects)

whole design matrix

(exercise)
- coceptual model = ?
- model matrix with block & treatment effects = ?
- 8 df. are used to estimate what?

True model: $Z = X_1\beta_1 + X_2\beta_2 + \varepsilon$
$H_1X_2\beta_2$ $(I-H_1)X_2\beta_2$

Fitted model: $Z = X_1\beta_1 + \varepsilon$ (X_2: block effects)

(a) If $X_1 = [\,I\ 1\ 2\ 3\,]$ ⇒ $H_1X_2\beta_2 = 0$. But, $\hat{\sigma}^2$ is larger than it should be.

(b) If $X_1 = [\,I\ 1\ 2\ 3\ 12\ 13\ 23\ 123\,]$ ⇒ $(I-H_1)X_2\beta_2 = 0$. But, $\hat{\beta}_{12}, \hat{\beta}_{13}, \hat{\beta}_{23}$ are biased

- $\{I, \mathbf{12}, \mathbf{13}, \mathbf{23}\}$ forms the **block defining contrast subgroup** for the 2^3 design in 4 blocks. For a more complicated example (2^5 design in 8 blocks), see textbook, p.196.

3 pseudo block factors b_1, b_2, b_3
⇒ 7 block effects: $b_1, b_2, b_3, b_1b_2, b_1b_3, b_2b_3, b_1b_2b_3$

Minimum Aberration Blocking Scheme

criterion

Only consider block schemes of the form: $B = \{I, b_1, b_2, b_1b_2\}$ ← a group
chosen from the columns in the model matrix of treatment effects
(Q: What is the best choice?)

- For any blocking scheme B and $1 \le i \le k$, let
  # of treatment factors

  $g_i(B)$ = number of i-factor effects that are confounded with block effects.

  ← # of i-factor (treatment) effects that are sacrified.

  also called a word of length i

  – Must require $g_1(B) = 0$ (because no main effect should be confounded with block effects).

- For any two blocking schemes B_1 and B_2, let

  $$r = \text{smallest } i \text{ such that } g_i(B_1) \neq g_i(B_2).$$

  wordlength pattern (WLP) of block scheme

  – If $g_r(B_1) < g_r(B_2)$, scheme B_1 is said to have *less aberration* than scheme B_2.

  – A blocking scheme B has *minimum aberration* (MA) if no other blocking schemes have less aberration than B.

  MA criterion $(g_1(B), g_2(B), \cdots, g_k(B))$ sequentially minimize

- The minimum aberration criterion is justified by the effect hierarchy principle.
- Minimum aberration blocking schemes are given in Table 4A.1 (textbook, p.207).
- Theory is developed under the assumption of no *block* $\times$ *treatment* interactions.

❖ **Reading**: textbook, 4.15