

2^k Designs in 2^q Blocks

k treatment factors, each 2 levels
 full factorial $\Rightarrow 2^k$ treatments (\Rightarrow need 2^k EUs)

one block factor, 2^q levels
 block size 2^{k-q} (\because total 2^k EUs)

- Example: Arranging a 2^3 design in 2 blocks (of size 4). Use the 123 column in Table 9 (LNp.5-37) to define the blocking scheme:

block size $2^{k-q} < 2^k$ treatments \Rightarrow incomplete blocking

block I if 123 = -, and block II if 123 = +.

★ conceptual model:
 $Z \sim \beta_0 + \text{block effects}$
 $\uparrow 2^q - 1$ parameters
 + all treatment effects + ϵ
 $\uparrow 2^k - 1$ parameters
 \Rightarrow no interaction btwn block & treatment factors

- The block effect estimate $\bar{y}(II) - \bar{y}(I)$ is identical to the estimate of the 123 interaction $\bar{y}(123 = +) - \bar{y}(123 = -)$. The block effect b and the interaction 123 are called **confounded**. Notationally,

aliases (Ch5)

$b = 123$

sacrificing

- In design matrix, column $b = (\text{column 1}) \times (\text{column 2}) \times (\text{column 3})$
- In model matrix, block effect column = 123 interaction column
- $\beta_b + \beta_{123}$ is jointly estimated

- By giving up the ability to estimate 123, this blocking scheme increases the precision in the estimates of main effects and 2fi's by arranging 8 runs in two homogeneous blocks.

a good choice?

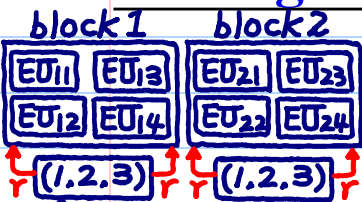
\because the block effect ① can reduce σ^2 ② is orthogonal to other treatment effects (their estimates not biased by block effect)

- Why sacrificing 123?

ans: Effect hierarchy principle.

123 is the least important effect among all the 8 factorial effects.

Arrangement of 2^3 Design in 2 Blocks



conceptual model: (Note, not interested in β_0)
 $Z = \beta_0 + \beta_b \chi_b + \beta_1 \chi_1 + \beta_2 \chi_2 + \beta_3 \chi_3 + \beta_{12} \chi_{12} + \beta_{13} \chi_{13} + \beta_{23} \chi_{23} + \beta_{123} \chi_{123} + \epsilon$
 block effect \uparrow
 $= \beta_0 + (\beta_b + \beta_{123}) \chi_b + \beta_1 \chi_1 + \beta_2 \chi_2 + \beta_3 \chi_3 + \beta_{12} \chi_{12} + \beta_{13} \chi_{13} + \beta_{23} \chi_{23} + \epsilon$

Table 9: Arranging a 2^3 Design in Two Blocks of Size Four (The 3 factors are denoted by 1, 2, and 3)

Run	Intercept	1	2	3	12	13	23	123	Block
1	+	-	-	-	+	+	+	-	I
2	+	-	-	+	+	-	-	+	II
3	+	-	+	-	-	+	-	+	II
4	+	-	+	+	-	-	+	-	I
5	+	+	+	-	-	-	+	+	II
6	+	+	-	+	-	+	-	-	I
7	+	+	+	-	+	-	-	-	I
8	+	+	+	+	+	+	+	+	II

In the model matrix, the 8 d.f. are used to estimate β_0 (intercept), $\beta_1, \beta_2, \beta_3, \beta_{12}, \beta_{13}, \beta_{23}, \beta_b + \beta_{123}$

a joint effect

model matrix (all treatment effects)

design matrix of treatment factors

combine whole design matrix

of possible columns: 7 \Rightarrow guarantee orthogonality

$\chi_b = \chi_{123}$

design matrix of block factor

of possible choices: $\binom{8}{4} = \frac{8!}{4!4!}$

Why might not be good choices? Ans. It would be better to have orthogonality.

A 2^3 Design in 4 Blocks

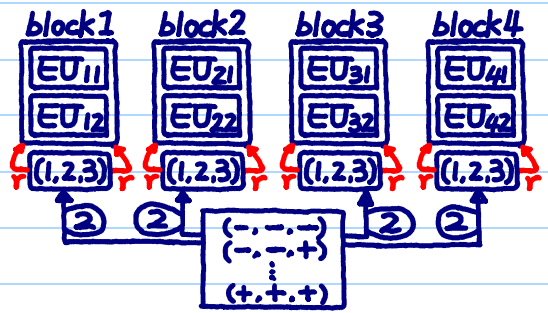
one block factor, 4 levels
block size 2 $\leftarrow = 2^2$

8 treatments each with 2 levels

- Similarly we can use $b_1 = 12$ and $b_2 = 13$ to define two independent blocking variables.

The 4 blocks I, II, III and IV are defined by

$b_1 = \pm$ and $b_2 = \pm$:



pseudo block factors

	b_1	
b_2	-	+
-	I	III
+	II	IV

	β_{b_1}	β_{b_2}	$\beta_{b_1 b_2}$
block	b_1	b_2	$b_1 b_2$
I	-	-	+
II	-	+	-
III	+	-	-
IV	+	+	+

This is not an interaction effect. It represents a block main effect.

A group structure in Algebra, with 2 generators b_1, b_2

a block factor with 4 levels needs 3 df. for its block effects.

Q: Why not sacrifice 123?

	$\{I, b_1, b_2, b_1 b_2\}$
123	1×23
	2×13
	3×12
identity element	12×3
	13×2
Intercept	23×1

- A 2^3 design in 4 blocks is given in Table 10 (LNp.5-39). Confounding relationships:

$b_1 = 12, b_2 = 13, b_1 b_2 = 12 \times 13 = 23 = 1^2 23$

Thus 12, 13 and 23 are confounded with block effects and thus sacrificed.

Note. In model matrix, for every columns X_i 's, we have $X_i^2 = I$

Arranging a 2^3 Design in 4 Blocks

Table 10: Arranging a 2^3 Design in Four Blocks of Size Two

		$\beta_{b_2} + \beta_{13}$	$\beta_{b_1 b_2} + \beta_{23}$						
	$\binom{7}{2}$ choices	b_1	b_2	$b_1 b_2$					
Run	I	1	2	3	12	13	23	123	block
1	+	-	-	-	+	+	+	-	IV
2	+	-	-	+	+	-	-	+	III
3	+	-	+	-	-	+	-	+	II
4	+	-	+	+	-	-	+	-	I
5	+	+	-	-	-	-	+	+	I
6	+	+	-	+	-	+	-	-	II
7	+	+	+	-	+	-	-	-	III
8	+	+	+	+	+	+	+	+	IV

of possible choices: $\binom{8}{2} = 28$

(exercise)
 • conceptual model = ?
 • model matrix with block & treatment effects = ?
 • 8 df. are used to estimate what?

True model: $Z = X_1 \beta_1 + X_2 \beta_2 + \epsilon$
 $H_1 X_2 \beta_2 + (I - H_1) X_2 \beta_2$

Fitted model: $Z = X_1 \beta_1 + \epsilon$
 $(X_2: \text{block effects})$

(a) If $X_1 = \begin{bmatrix} I & 12 & 13 \\ | & | & | \end{bmatrix} \Rightarrow \begin{cases} H_1 X_2 \beta_2 = 0 \\ \text{But, } \beta^2 \text{ is larger than it should be.} \end{cases}$

(b) If $X_1 = \begin{bmatrix} I & 12 & 13 & 12 & 13 & 23 & 123 \\ | & | & | & | & | & | & | \end{bmatrix} \Rightarrow \begin{cases} (I - H_1) X_2 \beta_2 = 0 \\ \text{But, } \hat{\beta}_{12}, \hat{\beta}_{13}, \hat{\beta}_{23} \text{ are biased} \end{cases}$

model matrix (treatment effects)

whole design matrix

- $\{I, 12, 13, 23\}$ forms the block defining contrast subgroup for the 2^3 design in 4 blocks. For a more complicated example (2^5 design in 8 blocks), see textbook, p.196.

3 pseudo block factors $b_1, b_2, b_3 \Rightarrow 7$ block effects: $b_1, b_2, b_3, b_1 b_2, b_1 b_3, b_2 b_3, b_1 b_2 b_3$

Minimum Aberration Blocking Scheme

Only consider block schemes of the form: ^{p. 5-40}

$$B = \{ I, b_1, b_2, b_1 b_2 \} \leftarrow \text{a group}$$

chosen from the columns in the model matrix of treatment effects
(Q: What is the best choice?)

↑ criterion

of treatment factors

- For any blocking scheme B and $1 \leq i \leq k$, let

also called a word of length i

$g_i(B)$ = number of i -factor effects that are confounded with block effects.

of i -factor (treatment) effects that are sacrificed.

- Must require $g_1(B) = 0$ (because no main effect should be confounded with block effects).

- For any two blocking schemes B_1 and B_2 , let

$$r = \text{smallest } i \text{ such that } g_i(B_1) \neq g_i(B_2).$$

wordlength pattern (WLP) of block scheme

- If $g_r(B_1) < g_r(B_2)$, scheme B_1 is said to have less aberration than scheme B_2 .

- A blocking scheme B has minimum aberration (MA) if no other blocking schemes have less aberration than B .

MA criterion $(g_1(B), g_2(B), \dots, g_k(B))$ sequentially minimize

- The minimum aberration criterion is justified by the effect hierarchy principle.
- Minimum aberration blocking schemes are given in Table 4A.1 (textbook, p.207).
- Theory is developed under the assumption of no block \times treatment interactions.

❖ Reading: textbook, 4.15