ANOVA Decomposition (Cont'd)

Table 26: Wood Experiment: Summarized data for whole plot analysis (LNp.5-48)

S_2; $S_2 = \mathcal{R}_2 \ominus \mathcal{R}_1$

	Rep 1	Rep 2	Rep 3	Total
a_1	181.1	224.7	219.0	624.8
a_2	168.0	191.0	128.8	487.8
Total	349.1	415.7	347.8	1112.6

$\sum_{j=1}^{J} y_{kij}$: (LNp.53) they are proportional to the projection of $\mathbf{y}$ onto the 6-dimensional space $\mathcal{R}_2$

$\|P_{\mathcal{R}_2}\mathbf{y}\|^2 = \sum_{\ell=1}^{n_W} k_W \bar{y}_\ell^{W2} = [\sum_{\ell=1}^{n_W} (\bar{y}_\ell^W \times k_W)^2]/k_W$

from data in LNp.48; only 6 obs. ⇒ total d.f. = 6 = n_W ↳ like a 2-way layout

$W_1 \oplus W_0$; W_0; $T_1 \oplus W_0 = S_1$

$P_{\mathcal{R}_2}\mathbf{y} = (\bar{y}^W_1, \ldots, \bar{y}^W_1, \ldots, \bar{y}^W_6, \ldots, \bar{y}^W_6)^T$ (k_W times), $P_{\mathcal{R}_1}\mathbf{y} = (\bar{y}^R_1, \ldots, \bar{y}^R_1, \bar{y}^R_2, \ldots, \bar{y}^R_2, \bar{y}^R_3, \ldots, \bar{y}^R_3)^T$ (k_R times)

(1) W_1 → $SS_A = (624.8^2 + 487.8^2)/12 - 1112.6^2/24 = 782.04$, (12 = 3×4, 24 = 6×4) check LNp.58

⊕

(2) T_1 → $SS_{\text{Rep}} = (349.1^2 + 415.7^2 + 347.8^2)/8 - 1112.6^2/24 = 376.99$, (8 = 2×4, 24 = 6×4)

⊕

(2) T_2 → $SS_{\text{whole}} = SS_{\text{Rep}\times A} = 398.37$, → $MS_{\text{whole}} = \frac{SS_{\text{whole}}}{\dim(T_2)}$ $\xrightarrow{\text{estimate}}$ $k_W\sigma_W^2 + \sigma_S^2$

$\|P_{\mathcal{R}_2}\mathbf{y}\|^2 - 1112.6^2/24 - 782.04 - 376.99$

$S_2 = \mathcal{R}_2 \ominus W_0$; (6) (1)

$SS_{\text{sub}} = 927.88 - SS_{\text{whole}} - SS_{\text{Rep}} = 152.52.$ (T_3, T_2, T_1)

$T_1 \oplus T_2 \oplus T_3 = V_K^{\perp}$

can use $\bar{y}_1^W, \ldots, \bar{y}_{n_W}^W$ to calculate these SS's

check Table 25 in LNp.57, $\|P_{V_K^\perp}\mathbf{y}\|^2$

Expected Mean Squares in ANOVA

By (L4)(L5) in LNp.55

	Source	Effect	df	E(Mean Squares)
$W_0 \oplus T_1 = S_1 \supset T_1$; $\dim(S_1) = n_R$	Replicate (or block)	τ_k (random effects)	$n-1$ ($= n_R - \dim(W_0)$)	$\sigma_S^2 + J\sigma_W^2 + IJ\sigma_\tau^2$
S_2: W_1	A	α_i (fixed effects)	$I-1$ ($= \dim(W_1)$)	$\sigma_S^2 + J\sigma_W^2 + \frac{nJ\sum_{i=1}^I \alpha_i^2}{I-1}$
S_2: T_2 (Rep×A)	Whole plot error	ε^W_{ki}	$(I-1)(n-1)$ ($= (n_W - n_R) - \dim(W_1)$)	$\sigma_S^2 + J\sigma_W^2$
S_3: W_2	B	β_j (fixed effects)	$J-1$ ($= \dim(W_2)$)	$\sigma_S^2 + \frac{nI\sum_{j=1}^J \beta_j^2}{J-1}$
S_3: W_3	$A \times B$	$(\alpha\beta)_{ij}$	$(I-1)(J-1)$ ($= \dim(W_3)$)	$\sigma_S^2 + \frac{n\sum_{i=1}^I\sum_{j=1}^J (\alpha\beta)_{ij}^2}{(I-1)(J-1)}$
S_3: T_3 (Rep×B)⊕(Rep×A×B)	Subplot error	ε^S_{kij}	$I(J-1)(n-1)$ ($= (N - n_W) - \dim(W_2) - \dim(W_3)$)	σ_S^2

$\dim(S_1 \oplus S_2) = n_W$

$e_1 = \sigma_S^2 + k_W\sigma_W^2 + k_R\sigma_R^2$

$E[\|P_{W_1}\boldsymbol{\varepsilon}\|^2/\dim(W_1)]$; $e_2 = \sigma_S^2 + k_W\sigma_W^2$; $\|P_{W_1}X\boldsymbol{\beta}\|^2/\dim(W_1)$

By (L4) in LNp.55 & $\sum_i \alpha_i = 0$

what if $\alpha_1 = \cdots = \alpha_I = 0$?

$\|P_{W_2}X\boldsymbol{\beta}\|^2/\dim(W_2)$; what if $\beta_1 = \cdots = \beta_J = 0$?

$\|P_{W_3}X\boldsymbol{\beta}\|^2/\dim(W_3)$; e_3; what if $\alpha\beta_{ij}$'s $= 0$?

- Proofs are similar to but more tedious than in one-way random effects model (LNp.3-33∼36).

Hypothesis Testing

By (L3) in LNp.54, (L5) in LNp.55

$$F_A = \frac{MS_A}{MS_{whole}} \Rightarrow H_0^1 : \alpha_1 = \cdots = \alpha_I,$$

$W_1 \subset S_2$, $T_2 \subset S_2$

$\hat{e}_2 = \sigma_S^2 + k_W \sigma_W^2$

under H_0^1, $P_{W_1} X\beta = 0$ $\Rightarrow SS_A \sim e_2 \chi^2_{dim(W_1)}$

under $H_A^1 \Rightarrow SS_A \sim e_2 \times$ noncentral χ^2

$$F_B = \frac{MS_B}{MS_{sub}} \Rightarrow H_0^2 : \beta_1 = \cdots = \beta_J,$$

$W_2 \subset S_3$, $T_3 \subset S_3$

under $H_0^1 \cup H_A^1$, $SS_{whole} \sim e_2 \chi^2_{dim(T_2)}$

$\hat{e}_3 = \hat{\sigma}_S^2$

apply similar argument as for H_0^1

Q: How to estimate σ_S^2, σ_W^2, σ_R^2?

$$F_{AB} = \frac{MS_{A\times B}}{MS_{sub}} \Rightarrow H_0^3 : (\alpha\beta)_{ij} = \text{constant},$$

$W_3 \subset S_3$, $T_3 \subset S_3$

apply similar argument as for H_0^1

$$i = 1, \ldots, I,\ j = 1, \ldots, J.$$

$\hat{e}_1 = \sigma_S^2 + k_W \sigma_W^2 + k_R \sigma_R^2$

$$F_{Rep} = \frac{MS_{Rep}}{MS_{whole}} \Rightarrow H_0^4 : \sigma_\tau = 0.$$

$T_1 \subset S_1$, $S_2 \supset T_2$, $\sigma_\tau \to \sigma_R$

What happens if τ_k's are treated as fixed effects? $\mathbb{R}^N = W_0 \oplus W_\tau \oplus W_1 \oplus \cdots \oplus W_3 \oplus T_2 \oplus \cdots \oplus S_3$, $W_\tau = T_1$, $\mathcal{R}_2$ (check LNp.53)

The ANOVA for one-way REM in LNp.3-33~34 cf. $F_{dim(T_1),dim(T_2)}$ under null $\frac{MS_{T_1}}{MS_{T_2}}$; $F_{dim(T_2),dim(T_3)}$ under null $\frac{MS_{T_2}}{MS_{T_3}}$

$MS_{T_1} \xrightarrow{est.} e_1$, $MS_{T_2} \xrightarrow{est.} e_2$, $MS_{T_3} \xrightarrow{est.} e_3$; $e_1 \geq e_2$ (when $\sigma_R^2 = 0$), $e_2 \geq e_3$ (when $\sigma_W^2 = 0$)

Correct ANOVA Analysis

Table25 (LNp.57) cf. Table 27: Correct ANOVA Table, Wood Experiment

Source	Degrees of Freedom	Sum of Squares	Mean Squares	F	p-value
S_1 T_1 → Replicate	$3-1=$ 2 (dim(S_1))	376.99	$k_R\sigma_R^2+k_W\sigma_W^2+\sigma_S^2$ ← 188.50	0.95	0.513
W_1 → A	$2-1=$ 1	782.04	782.04 (Note. 188.5 < 199.19)	3.93	0.186 ✗
T_2 → Whole plot error S_2	$3-1=$ 2 (dim(S_2))	398.37	$k_W\sigma_W^2+\sigma_S^2$ ← 199.19		
	927.88 = ⊕		Note. usually, $\sigma_W^2 > \sigma_S^2$		
W_2 → B	$4-1=$ 3	266.00	88.67	6.98	0.006 ✓
W_3 → $A \times B$	$(2-1)(4-1)=$ 3	62.79	20.93	1.65	0.230 ✗
S_3 T_3 → Subplot error	$18-3-3=$ 12 (dim(S_3))	152.52	$\hat{\sigma}_S^2$ ← 12.71		
$\mathcal{R}_3 \ominus W_0$ → Total	23	2038.72			

($k_R = 8$, $k_W = 4$)

Q: Why does A become insignificant?

Q: Why does B become significant?

3 (n_R) blocks, size 2 → $\mathcal{R}_2$ (dim $= n_W$) ← check Table 26 (LNp.61)

6 (n_W) whole plots, size 4

24 (N) sub-plots → $\mathcal{R}_3$ (dim $= N$)

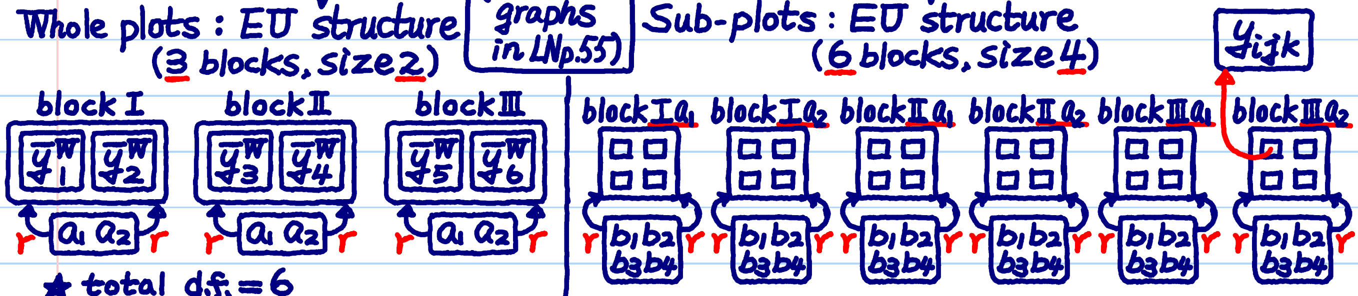
It is like treating $\tau_k + \alpha_i + \epsilon^W_{ki}$ (LNp.58) as a fixed block effect

Source	d.f.	SS	MS	F
Replicate	2	376.99	188.50	0.95
A	1	782.04	782.04	3.93
Whole plot error	2	398.37	199.19	
B	3	266.00	88.67	6.98
$A \times B$	3	62.79	20.93	1.65
Subplot error	12	152.52	12.71	
Total	23	2038.72		

Source	d.f.	SS	MS	F
6 blocks	5	1557.4	311.48	
B	3	266.00	88.67	6.98
$A \times B$	3	62.79	20.93	1.65
Subplot error	12	152.52	12.71	
Total	23	2038.72		

dim(S_2) $= n_W - n_R$ Randomized Block Design (check graphs in LNp.55)   dim(S_3) $= N - n_W$

Whole plots: EU structure (3 blocks, size 2)   Sub-plots: EU structure (6 blocks, size 4)



★ total d.f. = 6 = # of WP EUs n_W

★ # of blocks = 3 = # of replicates n_R

★ total d.f. = 24 = # of SP EUs N

★ # of blocks = 6 = # of WP EUs n_W

Analysis Results

- Only B is significant. ←cf.→ ANOVA results from 2-way layout (LNp.57) → only A is significant; $\text{cov}(\mathbf{y}) = \sigma^2 I$
- Explanation for discrepancy:

  $MS_{\text{whole}} = 199.19 \gg MS_{\text{Residual}} = 57.99 \gg MS_{\text{sub}} = 12.71.$

  - MS_{whole}: $k_W\sigma_W^2 + \sigma_S^2$ (T_2) ↔ A
  - MS_{Residual}: $\hat{\sigma}^2$ in 2-way layout; $V_K^{\perp} = T_1 \oplus T_2 \oplus T_3$
  - MS_{sub}: $\hat{\sigma}_S^2$ (T_3) ← B, A×B

  check Table 25 (LNp.57), Table 27 (LNp.64)

- To test $H_0^4: \sigma_\tau = 0$, use

$$\frac{MS_{\text{Rep}}}{MS_{\text{whole}}} = \frac{188.5}{199.19} = 0.95.$$

null dist. $F_{2,2}$ (T_1 / T_2); $< 1 \Rightarrow \hat{\sigma}_R^2 = 0$ (check LNp.3-37)

$\Rightarrow$ no significant difference between three replications.

- When does testing H_0^4 make sense? ← treat block effects as random effects and are interested in the population of all blocks. The n_R replicates should be a representative sample of the population.

❖ **Reading**: textbook, 3.9