Incorrect Model and Analysis

EUs are treated as in completely randomized design (homogeneous EUs) ↑ LNp.45-46

- Two-way layout model (treatment factors A and B with n replicates):

$$y_{ijk} = \eta + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \epsilon_{ijk},$$

different from the notations i, j, k in LNp.4-49~55

where $i = 1, \ldots, I$; $j = 1, \ldots, J$; $k = 1, \ldots, n$. (In the case, $I = 2$, $J = 4$, $n = 3$.)

★ conceptual model:

$$y \sim B_0 + \underbrace{A}_{(I-1)\text{ parameters}} + \underbrace{B}_{(J-1)\text{ parameters}} + \underbrace{A \times B}_{(I-1)(J-1)\text{ parameters}} + \underline{\varepsilon}$$

$\text{cov}(\underset{\sim}{\varepsilon}) = \underline{\sigma^2} I$

df. in $\hat{\sigma}^2$ $= IJn - IJ$

- ANOVA (table on LNp.4-57) shows that only factor A is significant; neither B nor $A \times B$ is significant.
- The model is wrong: A and B use *different* randomization schemes. The error component should be separated into two parts–the whole plot error and the subplot error. To test the significance of various effects, we need to compare their respective mean squares with two different error components.

Suppose the experiment is really from CRD. Some possible choices for $\text{Var}(\varepsilon_{ijk})$:

(a) $24\,\varepsilon^R, 24\,\varepsilon^W, 24\,\varepsilon^S \Rightarrow \text{cov}(\underset{\sim}{y}) = (\sigma_R^2 + \sigma_W^2 + \sigma_S^2) I$

(b) $1\,\varepsilon^R, 24\,\varepsilon^W, 24\,\varepsilon^S \Rightarrow \text{cov}(\underset{\sim}{y}) = (\sigma_W^2 + \sigma_S^2) I$

(c) $1\,\varepsilon^R, 1\,\varepsilon^W, 24\,\varepsilon^S \Rightarrow \text{cov}(\underset{\sim}{y}) = \sigma_S^2 I$

Incorrect ANOVA Table

Table 25: Incorrect ANOVA Table, Wood Experiment

Source	Degrees of Freedom	Sum of Squares	Mean Squares	F	p-value
A	1	782.04	782.04	13.49	0.002 ✓
B	3	266.00	88.67	1.53	0.245
$A \times B$	3	62.79	20.93	0.36	0.783
Residual	16	927.88	57.99		
Total	23	2038.72			

cf. Table 27 in LNp.64

$\hat{\sigma}^2$

- Only A is significant.

Correct Model for Split-plot Design

over-parameterized

$\mathcal{R}_2(6)$ ← n_W $\quad$ $\mathcal{R}_3 = \mathbb{R}^N(24)$ ← N $\quad$ check LNp 4-48, LNp.4-50, 4-51

$\mathcal{R}_1 = S_1(3)$ ← n_R $\quad$ $S_2(3)$ $\quad$ $S_3(18)$

- Model:

Under sum codings →

○ 規律 fixed effects $\quad$ √ 隨機 random effects or errors

$$y_{kij} = \overset{\circ W_0\,(1)}{\eta} + \overset{\checkmark T_1\,(2)}{\tau_k} + \overset{\circ W_1\,(1)}{\alpha_i} + \overset{\checkmark T_2\,(2)}{\varepsilon^W_{ki}} + \overset{\circ W_3\,(3)}{\beta_j} + \overset{\circ W_4\,(3)}{(\alpha\beta)_{ij}} + \overset{\checkmark T_3\,(12)}{\varepsilon^S_{kij}}, \quad (7)$$

where $\overset{A}{i} = 1,\ldots,I$; $\overset{B}{j} = 1,\ldots,J$; $\overset{Rep}{k} = 1,\ldots,n$. (In the case, $I = 2$, $J = 4$, $n = 3$.)

Terms related to the whole plot		Terms related to the subplot	
ε^R_k → τ_k ← 3	effect of kth replicate	β_j ← 4	jth main effect of B
α_i ← 2	ith main effect for A	$(\alpha\beta)_{ij}$ ← 8	(i, j)th interaction $A \times B$
ε^W_{ki} ← 6	(k,i)th whole plot error	ε^S_{kij} ← 24	(k,i, j)th subplot error

- y_{kij}: observation for the kth replicate of the ith level of factor A and the jth level of factor B.

★ conceptual model: ← check LNp.49

WP effect $\quad$ SP effects $\quad$ cov($\underset{\sim}{y}$)

$$y \sim \beta_0 + \underbrace{A}_{①} + \underbrace{B}_{③} + \underbrace{A\times B}_{③} + \underbrace{A_R\varepsilon^R}_{③} + \underbrace{A_W\varepsilon^W}_{⑥} + \underbrace{A_S\varepsilon^S}_{㉔}$$

E($\underset{\sim}{y}$)

$N = nIJ$
$n_R = n$, $k_R = IJ$
$n_W = nI$, $k_W = J$
$n_S = nIJ$, $k_S = 1$

Correct Model for Split-plot Design (Cont'd)

ε^R_k ← $n_R(3)$ τ_k's

- Treat τ_k as a **random** effect (because there are potentially many other possible replications), and $\tau_k \sim N(0, \sigma^2_\tau)$. ← σ_R^2

→ population

can be viewed as random block effects

- Treatment effects are *fixed* effects from a two-way layout, which include
  - α_i's, related to whole plots, and ← effects of whole plot factor
  - β_j's and $(\alpha\beta)_{ij}$'s, related to subplots. ← effects involving subplot factor

$n_W(6)$ ε^W's

- whole plot error: $\varepsilon^W_{ki} \sim N(0, \sigma^2_W)$, for testing α-effects.

→ $e_2 = \sigma_S^2 + k_W \sigma_W^2$

$n_S(24)$ ε^S's

- subplot error: $\varepsilon^S_{kij} \sim N(0, \sigma^2_S)$, for testing β- and $\alpha\beta$-effects.

→ $e_3 = \sigma_S^2$ $\quad$ Note $e_2 > e_3$

- Subplot error is usually *smaller* than whole plot error (i.e., $\sigma^2_S < \sigma^2_W$) because subplots are more homogeneous than whole plots.

← Why? ∵ nested (or split) structure

- Assume that τ_k's, ε^W_{kj}'s, and ε^S_{kij}'s are mutually independent.

ANOVA Decomposition

$W_1 = \text{span}\{X_1\}$

Note. All these SS's are independent. ← By (L6) in LNp.55

- Use the zero-sum constraints: → sum codings

$$\alpha_I = -(\alpha_1 + \cdots + \alpha_{I-1}) \xrightarrow{\text{e.g.}} \sum_{i=1}^{I} \alpha_i = \sum_{j=1}^{J} \beta_j = 0, \xleftarrow{\text{e.g.}} \beta_J = -(\beta_1 + \cdots + \beta_{J-1}) \to W_2 = \text{span}\{X_2\}$$

$$\alpha\beta_{iJ} = -(\alpha\beta_{i1} + \cdots + \alpha\beta_{i,J-1}) \xrightarrow{\text{e.g.}} \sum_{j=1}^{J} (\alpha\beta)_{ij} = 0, \quad \text{for } i = 1, \ldots, I,$$

$$\alpha\beta_{I,j} = -(\alpha\beta_{1j} + \cdots + \alpha\beta_{I-1,j}) \xrightarrow{\text{e.g.}} \sum_{i=1}^{I} (\alpha\beta)_{ij} = 0, \quad \text{for } j = 1, \ldots, J,$$

$\to W_3 = \text{span}\{X_3\}$

to break up the total sum of squares as a three-way layout with factors A, B, and Rep: $SS_{\underline{U}} = \|P_{\underline{U}} \underset{\sim}{y}\|^2$ A: 2 levels, B: 4 levels, Rep: 3 levels

treated as a 3-level factor

By △ in LNp 4-53

$$\underset{\mathbb{R}^N \ominus W_0}{SST} = \underset{T_1}{SS_{\text{Rep}}^{(2)}} + \underset{W_1}{SS_A^{(1)}} + \underset{W_2}{SS_B^{(3)}} + \underset{T_2}{SS_{\text{Rep}\times A}^{(2)}} + \underset{W_3}{SS_{A\times B}^{(3)}} + \underbrace{SS_{\text{Rep}\times B}^{(6)} + SS_{\text{Rep}\times A\times B}^{(6)}}_{T_3}.$$

- Define the sum of squares for the whole plot error SS_{whole} and the sum of squares for the subplot error SS_{sub} as:

W_i's, T_j's are the spaces generated by these effects | e.g., $\text{Rep}\times A \subset \mathcal{R}_2$ and $\text{Rep}\times A \perp W_0$, $\perp$ Rep $\}\mathcal{R}_1$, $\perp A$ (LNp.4-48)

$$(k_w\sigma_w^2 + \sigma_s^2) \times \dim(T_2) \xleftarrow{\text{est.}} SS_{\text{whole}} = SS_{\text{Rep}\times A}, \quad SS_{\text{sub}} = SS_{\text{Rep}\times B} + SS_{\text{Rep}\times A\times B}. \xrightarrow{\text{est.}} \sigma_s^2 \times \dim(T_3)$$

- ANOVA decomposition for the split-plot model:

By (L5) in LNp.55

$$SST = \underbrace{\overset{\sim e_1\chi^2_{\dim(T_1)}}{SS_{\text{Rep}}}}_{S_2} + \underbrace{SS_A + \overset{\sim e_2\chi^2_{\dim(T_2)}}{SS_{\text{whole}}}}_{} + \underbrace{SS_B + SS_{A\times B} + \overset{\sim e_3\chi^2_{\dim(T_3)}}{SS_{\text{sub}}}}_{S_3}.$$