$S_3 \to \mathcal{R}_3$, $S_2 \to \mathcal{R}_2$, $S_1 = \mathcal{R}_1$

– Thus, $\mathcal{R}_2 \supseteq \mathcal{R}_1$, $\mathcal{R}_3 \supseteq \mathcal{R}_1$; $\mathcal{R}_3 \supset \mathcal{R}_2 \cap \mathcal{R}_1^{\perp}$; $\mathcal{R}_1^{\perp} \supset \mathcal{R}_3 \cap \mathcal{R}_2^{\perp}$

	$\boldsymbol{v} \in S_1$	$\boldsymbol{v} \in S_2$	$\boldsymbol{v} \in S_3$
$\boldsymbol{\Sigma}_R$	$\boldsymbol{v} \to \boldsymbol{\Sigma}_R\boldsymbol{v} = (k_R\sigma_R^2)\boldsymbol{v}$	$\boldsymbol{v} \to \boldsymbol{\Sigma}_R\boldsymbol{v} = \boldsymbol{0}$	$\boldsymbol{v} \to \boldsymbol{\Sigma}_R\boldsymbol{v} = \boldsymbol{0}$
$\boldsymbol{\Sigma}_W$	$\boldsymbol{v} \to \boldsymbol{\Sigma}_W\boldsymbol{v} = (k_W\sigma_W^2)\boldsymbol{v}$	$\boldsymbol{v} \to \boldsymbol{\Sigma}_W\boldsymbol{v} = (k_W\sigma_W^2)\boldsymbol{v}$	$\boldsymbol{v} \to \boldsymbol{\Sigma}_W\boldsymbol{v} = \boldsymbol{0}$
$\boldsymbol{\Sigma}_S$	$\boldsymbol{v} \to \boldsymbol{\Sigma}_S\boldsymbol{v} = (\sigma_S^2)\boldsymbol{v}$	$\boldsymbol{v} \to \boldsymbol{\Sigma}_S\boldsymbol{v} = (\sigma_S^2)\boldsymbol{v}$	$\boldsymbol{v} \to \boldsymbol{\Sigma}_S\boldsymbol{v} = (\sigma_S^2)\boldsymbol{v}$
$\boldsymbol{\Sigma}$	$\boldsymbol{v} \to \boldsymbol{\Sigma}\boldsymbol{v} = (\boldsymbol{\Sigma}_R+\boldsymbol{\Sigma}_W+\boldsymbol{\Sigma}_S)\boldsymbol{v}$ $= (\sigma_S^2 + k_W\sigma_W^2 + k_R\sigma_R^2)\boldsymbol{v}$, $\equiv e_1$	$\boldsymbol{v} \to \boldsymbol{\Sigma}\boldsymbol{v}$ $= (\sigma_S^2 + k_W\sigma_W^2)\boldsymbol{v}$, $\equiv e_2$	$\boldsymbol{v} \to \boldsymbol{\Sigma}\boldsymbol{v}$ $= (\sigma_S^2)\boldsymbol{v}$, $\equiv e_3$

What happens if no balance condition → $\boldsymbol{\Sigma}$

why vanishes? check A & $\mathcal{E}^R$ in LNp.48

can easily derive Σ^{-1} or $\Sigma^{-1/2}$ from this (check LNp.48)

– Consequently, for $\boldsymbol{\Sigma}$,

$$\begin{cases} \text{eigenvalue:} & e_1, \quad e_2, \quad e_3 \\ & \updownarrow \quad \updownarrow \quad \updownarrow \\ \text{eigenspace:} & S_1, \quad S_2, \quad S_3 \end{cases}$$

only depend on the error structure in Y

stratum

$\because \sigma_R^2, \sigma_W^2, \sigma_S^2 > 0$

Note. $e_1 > e_2 > e_3 > 0$.

Note. $\mathbb{R}^N = S_1 \oplus S_2 \oplus S_3$, $S_1 \perp S_2 \perp S_3$

e.g. design with certain balance or orthogonal property. Check LNp.48

- Suppose that, under a *well-planned design*, we have

Intercept: $W_0 \subset S_1$, WP effects: $W_{i_1}, \cdots, W_{i_u} \subset S_2$, SP effects: $W_{i_{u+1}}, \ldots, W_{i_k} \subset S_3$

– Define $T_1 = S_1 \ominus W_0$ (規律)

$T_2 = S_2 \ominus (W_{i_1} \oplus \cdots \oplus W_{i_u})$

$T_3 = S_3 \ominus (W_{i_{u+1}} \oplus \cdots \oplus W_{i_k})$

error space 隨機

$\dim(T_1) = n_R - \dim(W_0)$

$\dim(T_2) = (n_W - n_R) - \sum_{l=1}^{u} \dim(W_{i_l})$

$\dim(T_3) = (N - n_W) - \sum_{l=u+1}^{k} \dim(W_{i_l})$

– Then,

隨機 when $\boldsymbol{\Sigma} = \sigma^2\boldsymbol{I}$ ($V_k = W_0 \oplus W_{i_1} \oplus \cdots \oplus W_{i_k}$, check LNp.49)

$*$ $T_1 \oplus T_2 \oplus T_3 = V_k^{\perp}$

1 error space $V_k^{\perp}$ split 3 error spaces T_1, T_2, T_3 when $\boldsymbol{\Sigma} = \boldsymbol{\Sigma}_R + \boldsymbol{\Sigma}_W + \boldsymbol{\Sigma}_S$

隨機 in S_1; 規律 in S_1; 規律 in S_2; 隨機 in S_2; 規律 in S_3; 隨機 in S_3

$$\circledast\ \mathbb{R}^N \begin{cases} = W_0 \oplus T_1 \oplus W_{i_1} \oplus \cdots \oplus W_{i_u} \oplus T_2 \oplus W_{i_{u+1}} \oplus \cdots \oplus W_{i_k} \oplus T_3 \\ = \quad S_1 \quad \oplus \quad S_2 \quad \oplus \quad S_3 \end{cases}$$

built from $E(Y) = X\beta$

built from $\text{cov}(Y) = \Sigma$

similar decomposition can be applied on the REM in LNp.3-31~38 (exercise)

$*$ $W_0 \perp T_1 \perp W_{i_1} \perp \cdots \perp W_{i_u} \perp T_2 \perp W_{i_{u+1}} \perp \cdots \perp W_{i_k} \perp T_3$

⊖ Hence,

P: orthogonal projection matrix (exercise)

$P_{\mathcal{R}_1}Y=?$ $P_{\mathcal{R}_2}Y=?$ $P_{\mathcal{R}_3}Y=?$ $P_{S_1}Y=?$ $P_{S_2}Y=?$ $P_{S_3}Y=?$

$P_{T_1} = P_{S_1} - P_{W_0}$

$P_{T_2} = P_{S_2} - P_{W_{i_1}} - \cdots - P_{W_{i_u}}$

$P_{T_3} = P_{S_3} - P_{W_{i_{u+1}}} - \cdots - P_{W_{i_k}}$

$\circledast$ $\boldsymbol{Y} = \boldsymbol{P}_{W_0}\boldsymbol{Y} + \boldsymbol{P}_{T_1}\boldsymbol{Y}$ ($= \boldsymbol{P}_{S_1}\boldsymbol{Y}$)

$+ \boldsymbol{P}_{W_{i_1}}\boldsymbol{Y} + \cdots + \boldsymbol{P}_{W_{i_u}}\boldsymbol{Y} + \boldsymbol{P}_{T_2}\boldsymbol{Y}$ ($= \boldsymbol{P}_{S_2}\boldsymbol{Y}$)

$+ \boldsymbol{P}_{W_{i_{u+1}}}\boldsymbol{Y} + \cdots + \boldsymbol{P}_{W_{i_k}}\boldsymbol{Y} + \boldsymbol{P}_{T_3}\boldsymbol{Y}$ ($= \boldsymbol{P}_{S_3}\boldsymbol{Y}$)

check LNp.50

Note. When $S_j = \mathcal{R}_j \ominus \mathcal{R}_{j-1}$,

$$\boldsymbol{P}_{S_j} = \boldsymbol{P}_{\mathcal{R}_j} - \boldsymbol{P}_{\mathcal{R}_{j-1}}$$

where $\because \mathcal{R}_1 = \text{span}\{A_R\}$

$\frac{1}{k_R}A_RA_R^T = \boldsymbol{P}_{\mathcal{R}_1} = \boldsymbol{A}_R(\boldsymbol{A}_R^T\boldsymbol{A}_R)^{-1}\boldsymbol{A}_R^T$

$\frac{1}{k_W}A_WA_W^T = \boldsymbol{P}_{\mathcal{R}_2} = \boldsymbol{A}_W(\boldsymbol{A}_W^T\boldsymbol{A}_W)^{-1}\boldsymbol{A}_W^T$

$I_N = \boldsymbol{P}_{\mathcal{R}_3} = \boldsymbol{A}_S(\boldsymbol{A}_S^T\boldsymbol{A}_S)^{-1}\boldsymbol{A}_S^T$

P_W: check LNp.2-34 & ★ in LNp.4-49

$\circledast$ $\boldsymbol{P}_{W_0}\boldsymbol{Y} \perp \boldsymbol{P}_{T_1}\boldsymbol{Y} \perp \boldsymbol{P}_{W_{i_1}}\boldsymbol{Y} \perp \cdots \perp \boldsymbol{P}_{W_{i_u}}\boldsymbol{Y} \perp \boldsymbol{P}_{T_2}\boldsymbol{Y} \perp \boldsymbol{P}_{W_{i_{u+1}}}\boldsymbol{Y} \perp \cdots \perp \boldsymbol{P}_{W_{i_k}}\boldsymbol{Y} \perp \boldsymbol{P}_{T_3}\boldsymbol{Y}$

partition of sums of squares

$\circledast$ $\|\boldsymbol{Y}\|^2 = \|\boldsymbol{P}_{W_0}\boldsymbol{Y}\|^2 + \|\boldsymbol{P}_{T_1}\boldsymbol{Y}\|^2 + \|\boldsymbol{P}_{W_{i_1}}\boldsymbol{Y}\|^2 + \cdots + \|\boldsymbol{P}_{W_{i_u}}\boldsymbol{Y}\|^2$
$+ \|\boldsymbol{P}_{T_2}\boldsymbol{Y}\|^2 + \|\boldsymbol{P}_{W_{i_{u+1}}}\boldsymbol{Y}\|^2 + \cdots + \|\boldsymbol{P}_{W_{i_k}}\boldsymbol{Y}\|^2 + \|\boldsymbol{P}_{T_3}\boldsymbol{Y}\|^2$

(assume ★ in LNp.52)

Note. Suppose $W_i = \text{span}\{X_i\}$. GLS = OLS iff $\text{span}\{X_i\} = \text{span}\{\Sigma^{-1}X_i\}$ (LM, LNp.6-2)

- Consider the sequential ANOVA: for $i = 1, \ldots, k$,

Under the linear model in LNp.49 → $H_0^{(i)}: \boldsymbol{\beta}_i = \mathbf{0}$ $(\omega_i = V_{i-1})$ vs. $H_A^{(i)}: \boldsymbol{\beta}_i \neq \mathbf{0}$ $(\Omega_i = V_i)$

and suppose $W_i = V_i \ominus V_{i-1} \subset S_j$. (e.g., $W_3 \subset S_2 \Rightarrow i=3 \rightarrow j=2$)

check (P1) in LNp.2-35

(L1) $SS_{W_i} \equiv RSS_{\omega_i} - RSS_{\Omega_i} = \|P_{W_i}Y\|^2$ and $df_{\omega_i} - df_{\Omega_i} = \dim(W_i) \equiv r_i$ (OLS)

(L2) $P_{W_i}Y = P_{W_i}X\boldsymbol{\beta} + P_{W_i}\boldsymbol{\varepsilon}$, where ($Y = X\beta + \varepsilon$)

check (P2) in LNp.2-35

Recall. For $v \in S_j$, $v \to \Sigma v = e_j v$

– $P_{W_i}X\boldsymbol{\beta} = P_{W_i}(X_i\boldsymbol{\beta}_i + \cdots + X_k\boldsymbol{\beta}_k) = P_{W_i}(u_i + \cdots + u_k)$

By (N1) in LNp.2-30 and $\varepsilon \sim N(0, \Sigma)$

– $P_{W_i}\boldsymbol{\varepsilon} \sim N(\mathbf{0}, P_{W_i}\Sigma P_{W_i}^T)$, where

Suppose $W_i = \text{span}\{B_i\} \Rightarrow$ columns of $B_i \in W_i \subset S_j$. Then, $P_{W_i} = B_i(B_i^TB_i)^{-1}B_i^T$, and

$$P_{W_i}\Sigma P_{W_i}^T = P_{W_i}e_jP_{W_i}^T = e_jP_{W_i}$$

$\Sigma P_{W_i}^T = \Sigma B_i(B_i^TB_i)^{-1}B_i^T = e_jB_i(B_i^TB_i)^{-1}B_i^T = e_jP_{W_i}$ ($\Sigma B_i = e_jB_i$)

the reason why we split $\mathbb{R}^N$ into 3 strata S_1, S_2, S_3

and $P_{W_i} = P_{W_i}^-$. ($P_{W_i}^3 = P_{W_i}$)

– Thus, $P_{W_i}Y \sim N(P_{W_i}(u_i + \cdots + u_k), e_jP_{W_i})$

$(e_jP_{W_i})(\frac{1}{e_j}P_{W_i})(e_jP_{W_i}) = e_jP_{W_i}$, $(e_jP_{W_i})^-$

(L3) $\|P_{W_i}Y\|^2 = Y^TP_{W_i}Y = (X\boldsymbol{\beta} + \boldsymbol{\varepsilon})^TP_{W_i}(X\boldsymbol{\beta} + \boldsymbol{\varepsilon})$

check (P3) in LNp.2-35

$$= \|P_{W_i}X\boldsymbol{\beta}\|^2 + 2(X\boldsymbol{\beta})^TP_{W_i}\boldsymbol{\varepsilon} + \boldsymbol{\varepsilon}^TP_{W_i}\boldsymbol{\varepsilon},$$

(= 0 under null)

By (N6) in LNp.2-30

where $\boldsymbol{\varepsilon}^TP_{W_i}\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^TP_{W_i}^TP_{W_i}P_{W_i}\boldsymbol{\varepsilon} = e_j(P_{W_i}\boldsymbol{\varepsilon})^T(P_{W_i}/e_j)(P_{W_i}\boldsymbol{\varepsilon}) \sim e_j\chi^2_{r_i}$

FYI. $\|P_{W_i}Y\|^2/e_j \sim$ noncentral $\chi^2_{r_i}(\|P_{W_i}X\boldsymbol{\beta}\|^2/e_j)$ ← noncentral parameter

(L4) $E(SS_{W_i}) = E(\|P_{W_i}Y\|^2) = \|P_{W_i}(u_i + \cdots + u_k)\|^2 + r_ie_j$ (cf.)

(L5) For the jth error space T_j $(\subset S_j)$, $j = 1, 2, 3$,

Intuition? check LNp.48 (cf.)

– $SS_{T_j} \equiv \|P_{T_j}Y\|^2 = Y^TP_{T_j}Y = Y^T\left(P_{S_j} - \sum_{W_i \subset S_j} P_{W_i}\right)Y$

$$= (Y^TP_{S_j}Y) - \sum_{W_i \subset S_j}(Y^TP_{W_i}Y) = \|P_{S_j}Y\|^2 - \sum_{W_i \subset S_j}\|P_{W_i}Y\|^2$$

Can use $\|P_{T_j}Y\|^2$, $j = 1, 2, 3$, to estimate e_1, e_2, e_3 or $\sigma_R^2, \sigma_W^2, \sigma_S^2$

(= 0, ∵ $T_j \subset V_k^\perp$)

– $P_{T_j}Y = P_{T_j}X\boldsymbol{\beta} + P_{T_j}\boldsymbol{\varepsilon} = P_{T_j}\boldsymbol{\varepsilon} \sim N(\mathbf{0}, e_jP_{T_j})$

$P_{T_1}\varepsilon = P_{T_1}(A_R\varepsilon^R + A_W\varepsilon^W + A_S\varepsilon^S)$, $P_{T_2}\varepsilon = P_{T_2}(A_W\varepsilon^W + A_S\varepsilon^S)$, $P_{T_3}\varepsilon = P_{T_3}(A_S\varepsilon^S)$

– $\|P_{T_j}Y\|^2 = \boldsymbol{\varepsilon}^TP_{T_j}\boldsymbol{\varepsilon} \sim e_j\chi^2_{\dim(T_j)}$ $\quad \frac{SS_{T_j}}{\dim(T_j)}$ unbiased estimator → e_j

– $E(SS_{T_j}) = E(\|P_{T_j}Y\|^2) = \dim(T_j)e_j = \left(\dim(S_j) - \sum_{W_i \subset S_j}\dim(W_i)\right)e_j$

(L6) Since all W_i's and T_j's are orthogonal subspaces of the eigenspaces of Σ,

e.g. $\text{cov}(P_{W_i}Y, P_{T_j}Y) = P_{W_i}\Sigma P_{T_j}^T = P_{W_i}e_jP_{T_j}^T = e_jP_{W_i}P_{T_j} = 0$

similar for $P_{W_i}\Sigma P_{W_\ell}$ and $P_{T_j}\Sigma P_{T_\ell}$.

$\Rightarrow$ all $P_{W_i}Y$'s and $P_{T_j}Y$'s are *independent* random vectors

$\Rightarrow$ all $\|P_{W_i}Y\|^2$'s and $\|P_{T_j}Y\|^2$'s are *independent* random variables

- A geometric view: (same structure as an LM with $\Sigma = \sigma^2 I$)