Statistical Model and Sequential ANOVA for the Wood Experiment

eg. $u=3$, $k=6$

A	C	AC	B	AB	BC
$(i_1,$	$i_2,$	$i_3)$	$i_4,$	$i_5,$	$i_6)$
$=(1,$	$3,$	6	$2,$	$4,$	$5)$
WP			SP		

- Consider a linear model $\underline{\boldsymbol{Y}} = \underline{\boldsymbol{X}\boldsymbol{\beta}} + \underline{\boldsymbol{\varepsilon}}$, where
  - $\underline{\boldsymbol{Y}} \in \mathbb{R}^{N}$: an $\underline{N}\times 1$ random vector and $\boldsymbol{Y} \sim \boldsymbol{N}(\boldsymbol{X\beta}, \boldsymbol{\Sigma})$ sample size $=24$
  - $\underline{\boldsymbol{X}} = \left[\underline{\boldsymbol{1}} \;\; \underbrace{\boldsymbol{X}_{i_1}^{W}\cdots\boldsymbol{X}_{i_u}^{W}}_{\text{whole-plot effects}} \;\; \underbrace{\boldsymbol{X}_{i_{u+1}}^{S}\cdots\boldsymbol{X}_{i_k}^{S}}_{\text{sub-plot effects}}\right]$ and
    - $\{i_1,\ldots,i_u,i_{u+1},\ldots,i_k\}$ is a permutation of $\{1,\ldots,k\}$ that represents the order of effect sequence in a sequential ANOVA
    - WLOG, assume $i_1<\cdots<i_u$ and $i_{u+1}<\cdots<i_k$
  - $\underline{\boldsymbol{\beta}} = \left[\boldsymbol{\beta}_0^T \;\; \boldsymbol{\beta}_{i_1}^T\cdots\boldsymbol{\beta}_{i_u}^T \;\; \boldsymbol{\beta}_{i_{u+1}}^T\cdots\boldsymbol{\beta}_{i_k}^T\right]^T$
  - $\underline{\boldsymbol{\varepsilon}} = \underline{\boldsymbol{A}_R\boldsymbol{\varepsilon}^R} + \underline{\boldsymbol{A}_W\boldsymbol{\varepsilon}^W} + \underline{\boldsymbol{A}_S\boldsymbol{\varepsilon}^S} \sim \boldsymbol{N}(\boldsymbol{0},\boldsymbol{\Sigma})$ (see next slide) and   cf. $\boldsymbol{X\beta}$ structure
    - $\boldsymbol{\varepsilon}^R \sim \boldsymbol{N}(\boldsymbol{0}, \sigma_R^2\boldsymbol{I}_{n_R})$, $\boldsymbol{\varepsilon}^W \sim \boldsymbol{N}(\boldsymbol{0}, \sigma_W^2\boldsymbol{I}_{n_W})$, $\boldsymbol{\varepsilon}^S \sim \boldsymbol{N}(\boldsymbol{0}, \sigma_S^2\boldsymbol{I}_{N})$
    - $\boldsymbol{\varepsilon}^R, \boldsymbol{\varepsilon}^W, \boldsymbol{\varepsilon}^S$ are independent
    - Then, $\boldsymbol{\Sigma} = cov(\boldsymbol{Y}) = cov(\boldsymbol{\varepsilon}) = cov(\boldsymbol{A}_R\boldsymbol{\varepsilon}^R) + cov(\boldsymbol{A}_W\boldsymbol{\varepsilon}^W) + cov(\boldsymbol{A}_S\boldsymbol{\varepsilon}^S)$

$$= \sigma_R^2\boldsymbol{A}_R\boldsymbol{I}_{n_R}\boldsymbol{A}_R^T + \sigma_W^2\boldsymbol{A}_W\boldsymbol{I}_{n_W}\boldsymbol{A}_W^T + \sigma_S^2\boldsymbol{A}_S\boldsymbol{I}_{N}\boldsymbol{A}_S^T$$

$$= \sigma_R^2\boldsymbol{A}_R\boldsymbol{A}_R^T + \sigma_W^2\boldsymbol{A}_W\boldsymbol{A}_W^T + \sigma_S^2\boldsymbol{A}_S\boldsymbol{A}_S^T$$

$$= \boldsymbol{\Sigma}_R + \boldsymbol{\Sigma}_W + \boldsymbol{\Sigma}_S$$

By (NI) in LNp.2-30

$\sigma_R^2, \sigma_W^2, \sigma_S^2$: variance components

correlations (≥ 0) btwn data:
- 0
- $\dfrac{\sigma_R^2}{\sigma_R^2+\sigma_W^2+\sigma_S^2}$
- $\dfrac{\sigma_R^2+\sigma_W^2}{\sigma_R^2+\sigma_W^2+\sigma_S^2}$

Define $\boldsymbol{\mu}_0, \boldsymbol{\mu}_1, \ldots, \boldsymbol{\mu}_k$, and $V_0, V_1, \ldots, V_k$, $W_0, W_1, \ldots, W_k$, as in LNp.2-31~32.

Assume orthogonality. Then, $W_{i_\ell} = \text{Span}\{X_{i_\ell}\}$ *

replication errors (random effects)   cf. fixed effects

$8=$ Check LNp.48

$$[a_{\ell m}^R] = \boldsymbol{A}_R = \begin{array}{l} k_R \text{ rows} \\ k_R \text{ rows} \\ k_R \text{ rows} \end{array} \begin{bmatrix} \boldsymbol{1} & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{1} & \cdots & \boldsymbol{0} \\ \vdots & & \ddots & \vdots \\ \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{1} \end{bmatrix}_{N\times n_R}, \quad \boldsymbol{\varepsilon}^R = \begin{bmatrix} \varepsilon_1^R \\ \vdots \\ \varepsilon_{n_R}^R \end{bmatrix}, \quad (n_R = 3)$$

$$a_{\ell m}^R = \begin{cases} 1, & \text{if } y_\ell \text{ is in the } m\text{th replicate,} \\ 0, & \text{otherwise} \end{cases}$$

$N = k_R \times n_R$ balance condition

$\sigma_R^2\boldsymbol{A}_R\boldsymbol{A}_R^T = \boldsymbol{\Sigma}_R = \sigma_R^2 \begin{bmatrix} \boldsymbol{J}_{k_R} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{J}_{k_R} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{J}_{k_R} \end{bmatrix}$,

a row in $\boldsymbol{\Sigma}_R$ is a column in $\boldsymbol{A}_R$

WP errors

$4=$

$$[a_{\ell m}^W] = \boldsymbol{A}_W = \begin{array}{l} k_W \text{ rows} \\ k_W \text{ rows} \\ \vdots \\ k_W \text{ rows} \end{array} \begin{bmatrix} \boldsymbol{1} & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{1} & \cdots & \boldsymbol{0} \\ \vdots & & \ddots & \vdots \\ \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{1} \end{bmatrix}, \quad \boldsymbol{\varepsilon}^W = \begin{bmatrix} \varepsilon_1^W \\ \vdots \\ \varepsilon_{n_W}^W \end{bmatrix}, \quad (6 = n_W)$$

$$a_{\ell m}^W = \begin{cases} 1, & \text{if } y_\ell \text{ is in the } m\text{th whole plot,} \\ 0, & \text{otherwise} \end{cases}$$

$N = k_W \times n_W$

$\sigma_W^2\boldsymbol{A}_W\boldsymbol{A}_W^T = \boldsymbol{\Sigma}_W = \sigma_W^2 \begin{bmatrix} \boldsymbol{J}_{k_W} & & \\ & \ddots & \\ & & \boldsymbol{J}_{k_W} \end{bmatrix}$,

SP errors

$$[a_{\ell m}^S] = \boldsymbol{A}_S = \boldsymbol{I}_N, \quad \boldsymbol{\varepsilon}^S = \begin{bmatrix} \varepsilon_1^S \\ \vdots \\ \varepsilon_N^S \end{bmatrix}.$$

$$a_{\ell m}^S = \begin{cases} 1, & \text{if } y_\ell \text{ is in the } m\text{th sub-plot,} \\ 0, & \text{otherwise} \end{cases}$$

Note. $k_R > k_W > 1$, $n_R < n_W < N$.

$\sigma_S^2\boldsymbol{A}_S\boldsymbol{A}_S^T = \boldsymbol{\Sigma}_S = \sigma_S^2\boldsymbol{I}_N$

- Eigenvalues and eigenspaces of Σ $(= \Sigma_R + \Sigma_W + \Sigma_S)$

$v \in \mathscr{R}_1^\perp \Rightarrow A_R^T v = 0$
$\Rightarrow \Sigma_R v \propto A_R A_R^T v = 0$

Recall principal component analysis

What happens if no balance condition

– For Σ_R, eigenvalue: $k_R\sigma_R^2$, 0; eigenspace: $\mathscr{R}_1 \equiv \text{span}\{A_R\}$, $\mathscr{R}_1^\perp$

check LNp.50

$\dim(\mathscr{R}_1) = n_R = \dim(\xi^R)$
$\dim(\mathscr{R}_1^\perp) = N - n_R$

$= \sigma_R^2 A_R I_{n_R} A_R^T = (A_R/\sqrt{k_R})(k_R\sigma_R^2 I_{n_R})(A_R/\sqrt{k_R})^T$

orthogonal matrix — diagonal matrix

– For Σ_W, eigenvalue: $k_W\sigma_W^2$, 0; eigenspace: $\mathscr{R}_2 \equiv \text{span}\{A_W\}$, $\mathscr{R}_2^\perp$

$\Sigma_W = (A_W/\sqrt{k_W})(k_W\sigma_W^2 I_{n_W})(A_W/\sqrt{k_W})^T$

$\dim(\mathscr{R}_2) = n_W = \dim(\xi^W)$
$\dim(\mathscr{R}_2^\perp) = N - n_W$

– For Σ_S, eigenvalue: σ_S^2; eigenspace: $\mathscr{R}_3 \equiv \text{span}\{A_S\} = \mathbb{R}^N$

$\Sigma_S = A_S(\sigma_S^2 I_N)A_S^T$

$\dim(\mathscr{R}_3) = N = \dim(\xi^S)$

– Because $\mathscr{R}_1 \subset \mathscr{R}_2 \subset \mathscr{R}_3$, we can define $S_1 = \mathscr{R}_1$ and for $j = 2, 3$,

$$S_j = \mathscr{R}_j \ominus \mathscr{R}_{j-1} = \mathscr{R}_j \cap \mathscr{R}_{j-1}^\perp .$$

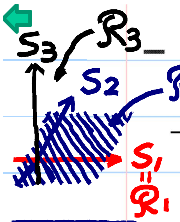
Check the example in LNp.48 for its $\mathscr{R}_1, \mathscr{R}_2, \mathscr{R}_3 \,|\, S_1, S_2, S_3$

Therefoer, we have

columns of $A_R \in$ span$\{A_W\}$
columns of $A_W \in$ span$\{A_S\}$
It happens because
WP nested in R
SP nested in WP
(check LNp4-46)

* $\mathscr{R}_j = S_1 \oplus \cdots \oplus S_j$
* $\mathbb{R}^N = S_1 \oplus S_2 \oplus S_3$ ($\mathscr{R}_3$)
* $S_1 \perp S_2 \perp S_3$

$\dim(S_1) = n_R$
$\dim(S_2) = n_W - n_R$
$\dim(S_3) = N - n_W$



– Thus,

$\mathscr{R}_2 \supseteq \mathscr{R}_1$, $\mathscr{R}_3 \supseteq \mathscr{R}_1$ | $\mathscr{R}_3 \supset \mathscr{R}_2 \cap \mathscr{R}_1^\perp$ | $\mathscr{R}_1^\perp \supset \mathscr{R}_3 \cap \mathscr{R}_2^\perp$

	$v \in S_1$	$v \in S_2$	$v \in S_3$
Σ_R	$v \to \Sigma_R v = (k_R\sigma_R^2)v$	$v \to \Sigma_R v = 0$	$v \to \Sigma_R v = 0$
Σ_W	$v \to \Sigma_W v = (k_W\sigma_W^2)v$	$v \to \Sigma_W v = (k_W\sigma_W^2)v$	$v \to \Sigma_W v = 0$
Σ_S	$v \to \Sigma_S v = (\sigma_S^2)v$	$v \to \Sigma_S v = (\sigma_S^2)v$	$v \to \Sigma_S v = (\sigma_S^2)v$
Σ	$v \to \Sigma v = (\Sigma_R+\Sigma_W+\Sigma_S)v = \underbrace{(\sigma_S^2 + k_W\sigma_W^2 + k_R\sigma_R^2)}_{\equiv e_1} v$	$v \to \Sigma v = \underbrace{(\sigma_S^2 + k_W\sigma_W^2)}_{\equiv e_2} v$	$v \to \Sigma v = \underbrace{(\sigma_S^2)}_{\equiv e_3} v$

What happens if no balance condition

can easily derive Σ^{-1} or $\Sigma^{-1/2}$ from this (check LNp.48)

– Consequently,

only depend on the error structure in Y

for Σ, eigenvalue: e_1, e_2, e_3; eigenspace: S_1, S_2, S_3

$\because \sigma_R^2, \sigma_W^2, \sigma_S^2 > 0$

Note. $e_1 > e_2 > e_3 > 0$.

Note. $\mathbb{R}^N = S_1 \oplus S_2 \oplus S_3$, $S_1 \perp S_2 \perp S_3$

stratum

e.g. design with certain balance or orthogonal property. Check LNp.48

- Suppose that, under a *well-planned design*, we have

Intercept: $W_0 \subset S_1$, WP effects: $W_{i_1}, \cdots, W_{i_u} \subset S_2$, SP effects: $W_{i_{u+1}}, \ldots, W_{i_k} \subset S_3$

– Define $T_1 = S_1 \ominus W_0$ 規律

error space 隨機

$T_2 = S_2 \ominus (W_{i_1} \oplus \cdots \oplus W_{i_u})$

$T_3 = S_3 \ominus (W_{i_{u+1}} \oplus \cdots \oplus W_{i_k})$

$\dim(T_1) = n_R - \dim(W_0)$
$\dim(T_2) = (n_W - n_R) - \sum_{l=1}^{u} \dim(W_{i_l})$
$\dim(T_3) = (N - n_W) - \sum_{l=u+1}^{k} \dim(W_{i_l})$

– Then,

$\ast$ $T_1 \oplus T_2 \oplus T_3 = V_k^\perp$ 隨機 when $\Sigma = \sigma^2 I$ ($V_k = W_0 \oplus W_1 \oplus \cdots \oplus W_k$)

1 error space $V_k^\perp$ $\xrightarrow{\text{split}}$ 3 error spaces T_1, T_2, T_3 when $\Sigma = \Sigma_R + \Sigma_W + \Sigma_S$

隨機 in S_1 規律 in S_1 規律 in S_2 隨機 in S_2 規律 in S_3 隨機 in S_3

similar decomposition can be applied on the REM in LNp.3-31~38 (exercise)

$$\circledast\ \mathbb{R}^N \begin{cases} = \underbrace{W_0 \oplus T_1}_{\|} \oplus \underbrace{W_{i_1} \oplus \cdots \oplus W_{i_u} \oplus T_2}_{\|} \oplus \underbrace{W_{i_{u+1}} \oplus \cdots \oplus W_{i_k} \oplus T_3}_{\|} \\ = \quad S_1 \quad \oplus \quad S_2 \quad \oplus \quad S_3 \end{cases}$$

built from $E(Y) = X\beta$

built from $cov(Y) = \Sigma$

$\ast$ $W_0 \perp T_1 \perp W_{i_1} \perp \cdots \perp W_{i_u} \perp T_2 \perp W_{i_{u+1}} \perp \cdots \perp W_{i_k} \perp T_3$

– Hence,

P: orthogonal projection matrix (exercise)

$P_{T_1} = P_{S_1} - P_{W_0}$

$P_{T_2} = P_{S_2} - P_{W_{i_1}} - \cdots - P_{W_{i_u}}$

$P_{T_3} = P_{S_3} - P_{W_{i_{u+1}}} - \cdots - P_{W_{i_k}}$

$\circledast$ $Y = \underbrace{P_{W_0}Y + P_{T_1}Y}_{\boxed{P_{S_1}Y =}}$

$\underbrace{+ P_{W_{i_1}}Y + \cdots + P_{W_{i_u}}Y + P_{T_2}Y}_{\boxed{P_{S_2}Y =}}$

$\underbrace{+ P_{W_{i_{u+1}}}Y + \cdots + P_{W_{i_k}}Y + P_{T_3}Y}_{\boxed{P_{S_3}Y =}}$

check LNp.50

$P_{\mathcal{R}_1}Y = ?$
$P_{\mathcal{R}_2}Y = ?$
$P_{\mathcal{R}_3}Y = ?$
$P_{S_1}Y = ?$
$P_{S_2}Y = ?$
$P_{S_3}Y = ?$

Note. When $S_j = \mathcal{R}_j \ominus \mathcal{R}_{j-1}$,

$$P_{S_j} = P_{\mathcal{R}_j} - P_{\mathcal{R}_{j-1}}$$

where $\because \mathcal{R}_1 = span\{A_R\}$

$\frac{1}{k_R}A_RA_R^T = P_{\mathcal{R}_1} = A_R(A_R^TA_R)^{-1}A_R^T$

$\frac{1}{k_W}A_WA_W^T = P_{\mathcal{R}_2} = A_W(A_W^TA_W)^{-1}A_W^T$

$I_N = P_{\mathcal{R}_3} = A_S(A_S^TA_S)^{-1}A_S^T$

$\circledast$ $P_{W_0}Y \perp P_{T_1}Y \perp P_{W_{i_1}}Y \perp \cdots \perp P_{W_{i_u}}Y$
$\perp P_{T_2}Y \perp P_{W_{i_{u+1}}}Y \perp \cdots \perp P_{W_{i_k}}Y \perp P_{T_3}Y$

P_W: check LNp.2-34 & ★ in LNp.4-49

partition of sums of squares

$\circledast$ $\|Y\|^2 = \|P_{W_0}Y\|^2 + \|P_{T_1}Y\|^2 + \|P_{W_{i_1}}Y\|^2 + \cdots + \|P_{W_{i_u}}Y\|^2$
$+ \|P_{T_2}Y\|^2 + \|P_{W_{i_{u+1}}}Y\|^2 + \cdots + \|P_{W_{i_k}}Y\|^2 + \|P_{T_3}Y\|^2$