Table 20: Wear Data, Tire Experiment

$t=4$, $b=4$, $k=3$

	Compound				
Tire	A	B	C	D	
1	238	238	279		$\bar{y}_{1\cdot}$
2	196	213		308	$\bar{y}_{2\cdot}$
3	254		334	367	$\bar{y}_{3\cdot}$
4		312	421	412	$\bar{y}_{4\cdot}$
$-\bar{y}_{\cdot\cdot}$	$\bar{y}_{\cdot 1}$	$\bar{y}_{\cdot 2}$	$\bar{y}_{\cdot 3}$	$\bar{y}_{\cdot 4}$	$\bar{y}_{\cdot\cdot}$
sum codings	τ_1	τ_2	τ_3	$(\tau_4=-\tau_1-\tau_2-\tau_3)$	

blocks

one-way ← $y \sim B_0 + \text{treatment} + \varepsilon \cdots (\triangle)$

$y \sim B_0 + \text{block} + \text{treatment} + \varepsilon \cdots (*)$

$H_1 X_b B_b$ $\quad X=[\mathbb{1}\ X_b\ X_t]$

A vs. B

$\bar{y}_{\cdot 1}-\bar{y}_{\cdot 2}$ ← 2 WBC, 1 BBC ← biased by block effect

- within block comparison (WBC): block effect can be eliminated
- between block comparison (BBC): block effect remains

λ

☆ 2 WBC, 1 BBC for A vs. C, A vs. D, B vs. C, B vs. D, C vs. D ⇐ balanced incomplete block design

$$E(\bar{y}_{\cdot 1}-\bar{y}_{\cdot\cdot}) = \tfrac{1}{3}(3\eta+\alpha_1+\alpha_2+\alpha_3+3\tau_1) - \tfrac{1}{12}(12\eta+3\alpha_1+3\alpha_2+3\alpha_3+3\alpha_4+3\tau_1+3\tau_2+3\tau_3+3\tau_4)$$

$$=(\tau_1-\bar{\tau})-\tfrac{1}{3}(\alpha_4-\bar{\alpha}) \neq \tau_1-\bar{\tau} \quad \leftarrow =0$$

Let $S_{\cdot j}=\frac{bk}{t}\bar{y}_{\cdot j}$ and $Q_j = S_{\cdot j}-\sum_i n_{ij}\bar{y}_{i\cdot}$ (# of treatment j in block i)

$$E(Q_1)=E(S_{\cdot 1}-\bar{y}_{1\cdot}-\bar{y}_{2\cdot}-\bar{y}_{3\cdot})$$

$$=(3\eta+3\tau_1+\alpha_1+\alpha_2+\alpha_3)-\tfrac{1}{3}(3\eta+3\alpha_1+\tau_1+\tau_2+\tau_3) -\tfrac{1}{3}(3\eta+3\alpha_2+\tau_1+\tau_2+\tau_4)-\tfrac{1}{3}(3\eta+3\alpha_3+\tau_1+\tau_3+\tau_4)$$

$$=(8/3)(\tau_1-\bar{\tau}) \Leftarrow (\lambda t/k)(\tau_1-\bar{\tau})$$

$$\Rightarrow E[(k/\lambda t)Q_j]=\tau_j-\bar{\tau} \Rightarrow \hat{\tau}_j=(k/\lambda t)Q_j$$

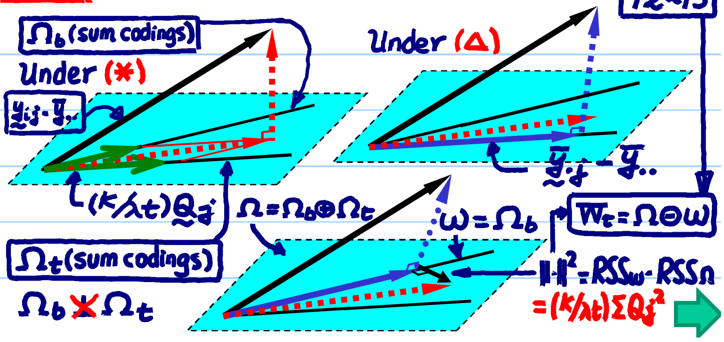
(exercise) $\quad cov(\hat{\beta})=(X^TX)^{-1}\sigma^2 \quad \hat{\beta}=(X^TX)^{-1}X^TY$ under $(*)$

$$cov\begin{pmatrix}\hat{\tau}_1\\ \vdots \\ \hat{\tau}_{t-1}\end{pmatrix} = \begin{pmatrix} v & & u \\ & \ddots & \\ u & & v\end{pmatrix}\sigma^2, \text{ where } u=\frac{-k}{\lambda t^2},\ v=\frac{k(t-1)}{\lambda t^2}$$

$$\Rightarrow var(\hat{\tau}_i-\hat{\tau}_j)=(2v-2u)\sigma^2=(2k/\lambda t)\sigma^2$$

σ^2 estimated from residuals of $(*)$ with d.f. $bk-b-t+1$

check LM, LNp. 3-12~13



Balanced Incomplete Block Design (BIBD)

Note. In the model matrix (under sum codings), block space (Ω_b) & treatment space (Ω_t) are not orthogonal.

- A BIBD has t treatments, and b blocks of size k, $t>k$, each treatment replicated r times, such that each pair of treatments appear in the *same* number (denoted by λ) of blocks. (In the wear experiment, $t=4$, $b=4$, $k=3$, $r=3$ and $\lambda=2$.)

# of WBC

ANOVA

cf. anova($y\sim B_0+B+T$) ✓ anova($y\sim B_0+T+B$) ✗

$\Omega: y\sim B_0+B+T$
$\omega: y\sim B_0+B$

source	d.f.	SS
B	$b-1$	□=?
T	$t-1$	□=?
r	$bk-b-t+1$	□=?

$RSS_\omega - RSS_\Omega=(k/\lambda t)\sum Q_j^2$

- Two basic relations:

(i) count # of EUs: $\quad b\times k = r\times t$, (i) $\quad$ (4×3, 3×4)

(ii) count # of WBCs in A vs. other factors: $\quad r\times(k-1)=\lambda\times(t-1)$. (ii) $\quad$ (3×2, 2×3)

$t>k$, $b>r$, $r>\lambda$, $rk>\lambda t$

multiple comparison

$(k/\lambda t)(Q_i-Q_j) \xrightarrow{\text{estimate}} \tau_i-\tau_j$

FYI Tukey's method can be applied to est'ors $\hat{\tau}_j$'s with covariance matrix → $\begin{pmatrix} v & u \\ u & v\end{pmatrix}$ (not independent)

$t_{ij}=(\hat{\tau}_i-\hat{\tau}_j)/s.e.(\hat{\tau}_i-\hat{\tau}_j)$, $\quad \sqrt{2k/\lambda t}\,\hat{\sigma}$

(Proof of (i) and (ii).)

- For given k, t and b, a BIBD may or may not exist. When it does not, either adjust the values of k, t, b to get a BIBD, or if not possible, find a partially balanced incomplete block design (PBIBD) (which is not covered in the book). Tables of BIBD or PBIBD in books like Cochran and Cox (1957).

construction of DM

❖ **Reading**: textbook, 3.8

Analysis of Covariance: Starch Experiment

- Data in Table 3.34 of textbook (p.129).

  Goal: To compare the three treatments (canna, corn, potato) for making starch film, y = break strength of film, covariate x = film thickness.

  x is not a factor (why?)

  Known that x affects y (thicker films are stronger); thickness cannot be controlled but are measured after films are made.

* response: strength
* treatment factor starch (qualitative) 3 levels - ca, co, po
* covariate thickness (quantitative)
* Exp'tal units: a film 49 EUs (heterogeneous, subject to the covariate)

It's a block factor if it's controllable

In modeling and analysis, it plays a role similar to block factor. cf.

There are cases in which covariate is treated as treatment factor

- **Question**: How to perform treatment comparisons by incorporating the effect of x?

- Model: $y_{ij} = \eta + \gamma \times x_{ij} + \tau_i + \varepsilon_{ij}$,

  linear effect of x → 1, k

  over-parameterized

  where $i = 1, \ldots, k$, $j = 1, \ldots, n_i$, and

  τ_i = ith treatment effect

  x_{ij} = covariate value,

  γ = regression coefficient for the x_{ij}

  ε_{ij} independent $N(0, \sigma^2)$.

* conceptual model: before exp't

$y \sim \beta_0 +$ thickness$_g$ + thickness$_g$ + ... + starch + ε

└ 2 parameters

⇒ main-effect-only model (why?)

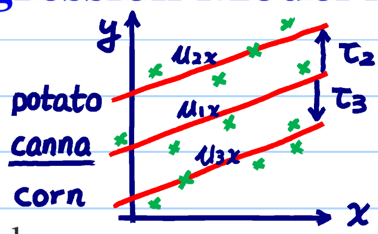
* Design matrix — Data matrix

starch	y	x
ca	791.7	7.7
⋮	⋮	⋮
ca	862.7	11.7
co	731.0	8.0
⋮	⋮	⋮
co	592.5	7.2
po	983.3	13.0
⋮	⋮	⋮
po	973.3	13.7

- **Special cases** of the model:
  1. When γ=0 (i.e., x_{ij} not available or no x effect), one-way layout.
  2. When τ_i=0 (no treatment effect), simple linear regression.

Regression Model Approach

add a constraint on $\tau_1, \ldots, \tau_k$, say, $\tau_1 = 0$



$\mu_{1,x} = E(y_{1,x}) = \eta + \gamma x$

$\mu_{2,x} = E(y_{2,x}) = \eta + \tau_2 + \gamma x$

$\mu_{3,x} = E(y_{3,x}) = \eta + \tau_3 + \gamma x$

η, γ: intercept & slope of $\mu_{1,x}$

$\tau_2 = \mu_{2,x} - \mu_{1,x}$

$\tau_3 = \mu_{3,x} - \mu_{1,x}$

- Model :

functional form

baseline (reference)

$$
\begin{aligned}
y_{1j} &= \eta + \quad\; + \gamma \times x_{1j} + \varepsilon_{1j}, \quad j = 1, \cdots, 13,\ i = 1 \quad \text{(canna)}\\
y_{2j} &= \eta + \tau_2 + \gamma \times x_{2j} + \varepsilon_{2j}, \quad j = 1, \cdots, 19,\ i = 2 \quad \text{(corn)} \qquad (6)\\
y_{3j} &= \eta + \tau_3 + \gamma \times x_{3j} + \varepsilon_{3j}, \quad j = 1, \cdots, 17,\ i = 3 \quad \text{(potato)}
\end{aligned}
$$

$\tau_1 = 0$ ⇒ treatment codings

where

τ_1 is set to zero (baseline constraint),

η = intercept,

γ = regression coefficient for thickness,

τ_2 = canna vs. corn, and

τ_3 = canna vs. potato.

(Write the model matrix for (6)).

(exercise) What if sum codings are used?

Matrix form $Y = X\beta + \varepsilon$

model matrix (MM $\overset{\text{cf.}}{\longleftrightarrow}$ DM)

$$
\begin{bmatrix} y_{11} \\ \vdots \\ y_{1n_1} \\ y_{21} \\ \vdots \\ y_{2n_2} \\ y_{31} \\ \vdots \\ y_{3n_3} \end{bmatrix}
=
\begin{bmatrix} 1 & 0 & 0 & x_{11} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & x_{1n_1} \\ 1 & 1 & 0 & x_{21} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 0 & x_{2n_2} \\ 1 & 0 & 1 & x_{31} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 1 & x_{3n_3} \end{bmatrix}
\begin{bmatrix} \eta \\ \tau_2 \\ \tau_3 \\ \gamma \end{bmatrix}
+ \varepsilon
$$

- Run regression analysis in the usual way.

Regression Analysis of Starch Experiment

Q: how to perform sequential ANOVA for this case? (cf drop-one test)

Table 21: Tests, Starch Experiment

	Effect	Estimate	Standard Error	t	p-value
η	intercept	158.261	179.775	0.88	0.38
γ	thickness	62.501	17.060	3.66	0.00
τ_2	canna vs. corn	−83.666	86.095	−0.97	0.34
τ_3	canna vs. potato	70.360	67.781	1.04	0.30
	corn vs. potato	154.026	107.762	1.43	0.16

H_0: $\tau_2 = \tau_3 \Leftarrow \omega$: merge τ_2, τ_3 columns in X (MM)

It shows the necessarity to to add the covariate, which plays a role similar to block factor in the analysis, but usually not orthogonal to treatment factor.

Ω: model (6) in LNp.41

ω: model (6) in LNp.41 with τ_2 setting to be 0

both models contain the term "γx"

- In the table, corn vs. potato $= \hat{\tau}_3 - \hat{\tau}_2 = 70.360 - (-83.666) = 154.026$.

  $\mu_{3.x} - \mu_{2.x} = (\mu_{3.x} - \mu_{1.x}) - (\mu_{2.x} - \mu_{1.x}) = \tau_3 - \tau_2$

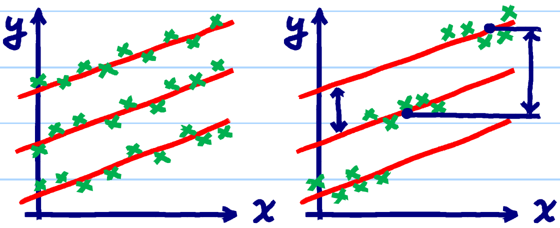
- No pair of film types has any significant difference after adjusting for thickness effect. (So, how should the choice be made between the three film types?) Most of the variation is explained by the covariate thickness.

"no significant difference" can also useful information for making decision.

Q: What if we fit the model: $y \sim \beta_0 + \text{starch} + \varepsilon$?

$Y = X_1\beta_1 + X_2\beta_2 + \varepsilon$

(B) $H_1X_2\beta_2$ $(I-H_1)X_2\beta_2$ (W)



Multiple Comparisons

(1) obtain $t_{ij} = \dfrac{\hat{\mu}_{jx} - \hat{\mu}_{ix}}{s.e.(\hat{\mu}_{jx} - \hat{\mu}_{ix})}$

(2) determine critical value (or p-value)

- $Var(\hat{\tau}_3)$ and $Var(\hat{\tau}_2)$ can be obtained from regression output Table 21 (LNp.4-42).

- From (1.33) of textbook (p.22),

  $Var(\hat{\beta}) = \sigma^2 (X^T X)^{-1}$.

  $\sigma^2 \begin{bmatrix} \ddots & & & \\ & (X^TX)^{-1}_{ii} \; (1) & (X^TX)^{-1}_{ij} \; (3) & \\ & & (X^TX)^{-1}_{jj} \; (2) & \\ & & & \ddots \end{bmatrix}$

  t_{12}, t_{13}

- Using this, for H_0: $\tau_2 = \tau_3$ (i.e., $\tau_3 - \tau_2 = 0$), the variance of $\hat{\tau}_3 - \hat{\tau}_2$ can be found as

  $t_{23} = \dfrac{(\hat{\tau}_3 - \hat{\tau}_2) - 0}{s.e.(\hat{\tau}_3 - \hat{\tau}_2)}$

$$Var(\hat{\tau}_3 - \hat{\tau}_2) = \underset{(2)}{Var(\hat{\tau}_3)} + \underset{(1)}{Var(\hat{\tau}_2)} - 2\,\underset{(3)}{Cov(\hat{\tau}_3, \hat{\tau}_2)}.$$

- The degrees of freedom for the t-statistic is same as that of the residuals. The p-values for the three tests are given in Table 21 (LNp.4-42).

  for specific (i,j)

- For simultaneous testing, use adjusted p-values (LNp.4-24).

  for all (i,j)'s

ANCOVA Table ← Apply sequential ANOVA to the model with covariate

↑ Analysis of covariance

treated as block factor and usually not orthogonal to treatment factor.

anova($y \sim B_0$ + thickness$_\ell$ + starch)

(M1) $y \sim B_0$

(M2) $y \sim B_0$ + thickness$_\ell$

(M3) $y \sim B_0$ + thickness$_\ell$ + starch

Note. covariate must appear before treatment factor

Table 22: ANCOVA Table, Starch Experiment

Source	Degrees of Freedom	Sum of Squares	Mean Squares	F
thickness	1	2553357	2553357	94.19
starch	2	56725	28362	1.05
residual	45	1219940	27110	

Ω: (M2), ω: (M1) → thickness

Ω: (M3), ω: (M2) → starch

not orthogonal

df. of residuals under (M3)

the t-value in Table 21 (LNp.42) — cf.

not significant

(exercise) Q. What if we use anova($y \sim B_0$ + starch + thickness$_\ell$)?

Note. It is possible that we are not interested in "starch" (↔ block), but are interested in the coefficient of "thickness" (↔ treatment)

❖ **Reading**: textbook, 3.10

Example of Split-plot Design: Wood Experiment

- Experiment objective : to study the water resistant property of wood. → response y
- Two treatment factors:
  - A — wood pretreatments: a_1, a_2; → qualitative, 2 levels
  - B — types of stain: b_1, b_2, b_3, b_4. → qualitative, 4 levels
- Completely randomized design: randomly apply the 8 level combinations of A and B to 8 wood panels, such as in Table 23.
- Problem: inconvenient to apply the pretreatments to a small wood panel.

✡ Exp'tal unit : a small wood panel

homogeneous → EU$_1$ • • • EU8

completely randomized → r (a$_1$,b$_1$) ⋯ (a$_2$,b$_4$) r

(check 2-way layout in LNp.4-15)

✡ Design matrix

pretreatment	stain	y
a_1	b_1	□⋯□
a_1	b_2	□⋯□
a_1	b_3	⋮
a_1	b_4	
a_2	b_1	
a_2	b_2	
a_2	b_3	□⋯□
a_2	b_4	□⋯□

Data matrix

of $\mathcal{E}$'s = # of EUs

✡ conceptual model : $y \sim B_0 + A + B + A{\times}B + \mathcal{E}$

1 parameter, 3 parameters, 3 parameters

EUs → y

Q: What does $\mathcal{E}$ represent?

✡ $\mathrm{cov}(y) = \sigma^2 I$

Q: How many times pretreatments (a_1 or a_2) are applied?

Ans: 8 times → 8 EUs

Table 23: Completely Randomized Version of the Wood Experiment

Run (EUs)	1	2	3	4	5	6	7	8
Pretreatment (A)	a_1	a_2	a_2	a_1	a_2	a_1	a_1	a_2
Stain (B)	b_2	b_4	b_1	b_1	b_3	b_4	b_3	b_2

← # of distinct level combinations of A and B