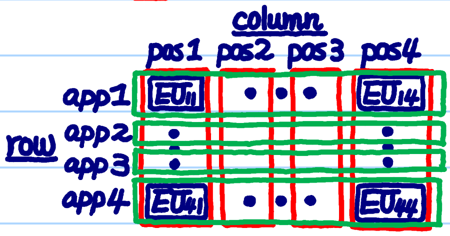
Latin Square Design : Wear Experiment

- **Wear Experiment :** Testing the abrasion resistance of rubber-covered fabric,
  - ★ response y: loss in weight over a period of time
  - ★ one treatment factor : material type A, B, C, D, → qualitative, 4 levels
  - ★ two blocking factors (crossing): → row-column structure
    (1) four positions (1, 2, 3, 4) on the tester, → qualitative, 4 levels, block size=4
    (2) four applications (1, 2, 3, 4; four different times for setting up the tester) → qualitative, 4 levels, block size=4

do not work for block factors with nesting structure.

check LNp.4-15

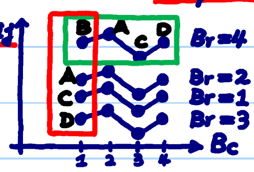
★ Exp'tal units : a position at an application (16 EUs : hetrogeneous)



Assume block factors have only ME

block effects → b_{ij}

within block comparison

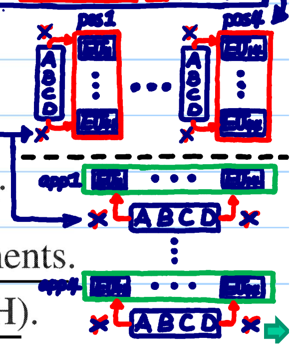


df. of the residuals in this RBD $=(k-1)^2 \geq k-1$ (if $k\geq 2$) ⇒ may accommodate one more k-level factor

- ◉ **Latin square (LS) design of order k :** Each of the k Latin letters (i.e., treatments) appears once in each of k rows and once in each of k columns. (block factor)

RBD → $(k!)^k$ cf. $(k!)^2$ ← level permutation of the three factors

more restriction on randomization than RBD



- It is an extension of RBD to accommodate *two* blocking factors. (← 1 block factor, 1 treatment factor)
- ◉ Randomization applied to assignments to rows, columns, treatments.
  (Collection of Latin Square Tables given in Appendix 3A of WH).

Wear Experiment : Design and Data

Table 13: Latin Square Design (columns correspond to positions, rows correspond to applications and Latin letters correspond to materials), Wear Experiment

Application (row)	Position (column) 1	2	3	4
1	C	D	B	A
2	A	B	D	C
3	D	C	A	B
4	B	A	C	D

Table 14: Weight Loss Data, Wear Experiment

Application	Position 1	2	3	4
1	235	236	218	268
2	251	241	227	229
3	234	273	274	226
4	195	270	230	225

Compare τ_A vs. τ_B (within block comparison)
$\bar{y}_A - \bar{y}_B$
column, row] block effects are removed in residuals
It's achieved because each treatment levels appear equally often in rows and columns

★ block factor | Design Matrix | treatment factor | randomization ⇔ level permutation

Data Matrix

y	Pos	App	Mat
235	1	1	C
⋮	1	2	A
	1	3	D
	1	4	B
	2	1	D
	2	2	B
	2	3	C
	2	4	A
	3	1	B
	3	2	D
	3	3	A
	3	4	C
	4	1	A
	4	2	C
⋮	4	3	B
225	4	4	D

full factorial design ← cf. → fractional factorial design

any 2 columns in design matrix are said to be orthogonal, if all level combinations of the 2 columns appear equally often

How to construct? → orthogonal array (OA) of strength 2 (textbook, p.323)

# of different level combinations of the three factors $= 4\times4\times4 = 64$. But, only 16 of them appear in the design matrix.

say, (1,1,A)? ← (Q : Can we perform all 64 level combinations?)

Model for Latin Square Design

(k-1) parameters

Conceptual model: y ~ β0 + row + column + treatment + ε ← main-effect-only model ⇒ no interaction (Why?)

before exp't

sum codings → W_r W_c W_t ⇒ $W_0 \perp W_r \perp W_c \perp W_t \perp \Omega^\perp$ ← Why?

- Model:

over-parameterized → $$y_{ijl} = \eta + \alpha_i + \beta_j + \tau_l + \varepsilon_{ijl},$$ (k) (k) (k)

∵ ① DM is an OA of strength 2, ② main-effect-only model ③ sum codings

where $i, j, l = 1, \ldots, k$, and

Note. l is a function of (i,j) → l = Latin letter in the (i, j) cell of the LS,

α_i = ith row effect,

β_j = jth column effect,

τ_l = lth treatment (i.e., Latin letter) effect,

ε_{ijl} are independent $N(0,\sigma^2)$.

Note. distinction of orthogonality in design & model matrices

Note. If there exist interactions btwn the 2 block factors (or in the nesting case) ← LNp.15 ⇒ conceptual model: y ~ β0 + row + column + row:column + treatment + ε ← $(k-1)^2$ parameters ⇒ can treat the 2 block factors as 1 block factor with k^2 levels ⇒ But, in LS, no d.f. left for treatment. ⇒ need to have multiple EUs in each of k^2 blocks. ⇒ Then, can use {RBD or BIBD

- There are only k^2 values in the triplet (i, j, l) dictated by the particular LS; ← Note. not k^3 this set is denoted by S.

Sum codings

$$y_{ijl} = \hat{\eta} + \hat{\alpha}_i + \hat{\beta}_j + \hat{\tau}_l + r_{ijl}$$

$$y = \underbrace{\bar{y}_{\cdots}}_{P_{W_0}(y)} + \underbrace{(\bar{y}_{i\cdot\cdot} - \bar{y}_{\cdots})}_{P_{W_r}(y)} + \underbrace{(\bar{y}_{\cdot j\cdot} - \bar{y}_{\cdots})}_{P_{W_c}(y)} + \underbrace{(\bar{y}_{\cdot\cdot l} - \bar{y}_{\cdots})}_{P_{W_t}(y)}$$

∵ orthogonality

$$+ (y_{ijl} - \bar{y}_{i\cdot\cdot} - \bar{y}_{\cdot j\cdot} - \bar{y}_{\cdot\cdot l} + 2\bar{y}_{\cdots}),$$

- ANOVA decomposition: similar formula (see (3.40) of WH)

ANOVA for Latin Square Design

anova(y ~ β0 + row + column + treatment) ← In sequential ANOVA, block factors should always be put before treatment factor

Table 15: ANOVA Table for Latin Square Design

	Source	Degrees of Freedom	Sum of Squares	MS = ?	E(MS) = ?
W_r	row	$k-1$	$k\sum_{i=1}^k (\bar{y}_{i\cdot\cdot} - \bar{y}_{\cdots})^2$		
W_c	column	$k-1$	$k\sum_{j=1}^k (\bar{y}_{\cdot j\cdot} - \bar{y}_{\cdots})^2$		
W_t	treatment	$k-1$	$k\sum_{l=1}^k (\bar{y}_{\cdot\cdot l} - \bar{y}_{\cdots})^2$		
$\Omega^\perp$	residual	$(k-1)(k-2)$	$\sum_{(i,j,l)\in S} (y_{ijl} - \bar{y}_{i\cdot\cdot} - \bar{y}_{\cdot j\cdot} - \bar{y}_{\cdot\cdot l} + 2\bar{y}_{\cdots})^2$		
$\mathbb{R}^N \ominus W_0$	total	$k^2 - 1$	$\sum_{(i,j,l)\in S} (y_{ijl} - \bar{y}_{\cdots})^2$		

$\Omega \ominus W_0$ = $W_r \oplus W_c \oplus W_t$

$\|P_{W_i}(y)\|^2$

use (P4) in LNp.2-36

Ω_t: y~row+column+treatment+ε
ω_t: y~row+column+ε

$\hat{\sigma}^2$ ← estimated from an RSS not containing block variation

β0

$k^2-1-3(k-1) = (k-1)(k+1-3)$
$= (k-1)^2 - (k-1)$ ← d.f. of residuals in RBD

Table 16: ANOVA Table, Wear Experiment

Source	Degrees of Freedom	Sum of Squares	Mean Squares	F
application	3	986.5	328.833	5.37
position	3	1468.5	489.500	7.99
material	3	4621.5	1540.500	25.15
residual	6	367.5	61.250	

4-1

largest SS

$\hat{\sigma}^2$

F-Test and Multiple Comparisons

F-test & multiple comparison for LSD use the same principles as in RBD, except that 2 block factors in the former while 1 in the latter

Ω_t: y~row+column+treatment
ω_t: y~row+column

- $H_0 : \tau_1 = \cdots = \tau_k$, can be tested by using the F-statistic

$$F = \frac{SS_t/(k-1)}{SS_r/(k-1)(k-2)},$$

RSS$_{\omega_t}$−RSS$_{\Omega_t}$ → SS_t; $(k-1)$ ← df$_{\omega_t}$−df$_{\Omega_t}$; RSS$_{\Omega_t}$ → SS_r; $(k-1)(k-2)$ ← df$_{\Omega_t}$

Note. ① $\bar{y}_{\cdot\cdot j}-\bar{y}_{\cdot\cdot i}$ is not biased by blocks (∵ orthogonality) ② $\hat{\sigma}^2$ = RSS$_{\Omega_t}$/df$_{\Omega_t}$ does not contain block variation.

The F-test rejects H_0 at level α if $F > F_{k-1,(k-1)(k-2),\alpha}$.

$Y = X_1\beta_1 + X_2\beta_2 + \varepsilon$
① $\cancel{H_1X_2\beta_2}$ ② $\cancel{(I-H_1)X_2\beta_2}$

- If H_0 is rejected, multiple comparisons of the τ_j should be performed. t-statistics for making multiple comparisons :

$H_0^{ij}: \tau_i = \tau_j$ $(\tau_j - \tau_i = 0)$
– for specific (i,j) → use t-dist. for null.
– for all (i,j)'s

$$t_{ij} = \frac{\bar{y}_{\cdot\cdot j} - \bar{y}_{\cdot\cdot i} - 0}{\hat{\sigma}\sqrt{1/k + 1/k}},$$

$\hat{\tau}_j - \hat{\tau}_i$ → $\bar{y}_{\cdot\cdot j} - \bar{y}_{\cdot\cdot i}$

$\boxed{Var(\bar{y}_{\cdot\cdot j}-\bar{y}_{\cdot\cdot i}) = \sigma^2/k + \sigma^2/k}$

where $\hat{\sigma}^2$ is the mean square error in the ANOVA table. ← $\boxed{\hat{\sigma}^2 = \frac{SS_r}{(k-1)(k-2)}}$

- At level α, the Tukey multiple comparison method identifies "treatments i and j as different" if

$$|t_{ij}| > \frac{1}{\sqrt{2}} q_{k,(k-1)(k-2),\alpha}.$$

Analysis Results

- The p-values for application (row) and position (column) are 0.039 (=$Prob(F_{3,6} > 5.37)$) and 0.016 (=$Prob(F_{3,6} > 7.99)$), respectively. This indicates that blocking is important.

- The treatment factor (material) has the most significance as indicated by a p-value of 0.0008 (= $Prob(F_{3,6} > 25.15)$).

- With k=4 and $(k-1)(k-2)$=6, the critical value for the Tukey multiple comparison method is

$$\frac{1}{\sqrt{2}} q_{4,6,0.05} = \frac{4.90}{\sqrt{2}} = 3.46$$

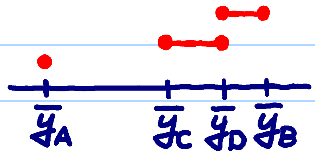
at the 0.05 level.

- By comparing the multiple comparisons t-statistics given in Table 17 (LNp.4-33) with 3.46, material A and B, A and C, A and D and B and C are identified as different at 0.05 level.

Multiple Comparisons Tables

Table 17: Multiple Comparison t-statistics, Wear Experiment

A vs. B	A vs. C	A vs. D	B vs. C	B vs. D	C vs. D
−8.27	−4.34	−6.37	3.93	1.90	−2.03



anova($y \sim \beta_0$+treatment) ← treated as one-way layout

Table 18: ANOVA Table (Ignoring Blocking), Wear Experiment ←cf.→ Table 16 (LNp.30)

Source	Degrees of Freedom	Sum of Squares	Mean Squares	F
material	same ←cf.→ 3	4621.5	1540.500	6.55
residual	12	2822.5	235.21 ← $\hat{\sigma}^2$ ← estimated from an RSS containing block variation	

6 ←cf. (12)   (2822.5) = 986.5 + 1468.5 + 367.5   (235.21) cf.→ 61.25

- Effectiveness of blocking:
  - With blocking, $Pr(F_{3,6} > 25.15) = 0.0008$.
  - Without blocking, $Pr(F_{3,12} > 6.55) = 0.007$.

  become less significant when block factors are significant.

  Therefore blocking can make a difference in decision making if treatment effects are smaller.

Q. What if some block factor is found insignificant.

❖ **Reading**: textbook, 3.6

Graeco-Latin Square Design

level combinations of Latin & Greek

- Two Latin squares are *orthogonal* if each pair of letters (Latin and Greek) appears once in the two squares, when superimposed. The super-imposed square is called a *Graeco-Latin square*.

or appears equally often

row	column 1	column 2	column 3
1	$A\alpha$	$B\beta$	$C\gamma$
2	$B\gamma$	$C\alpha$	$A\beta$
3	$C\beta$	$A\gamma$	$B\alpha$

⟵

orthogonal

1st LS

row	column 1	column 2	column 3
1	A	B	C
2	B	C	A
3	C	A	B

+

2nd LS

row	column 1	column 2	column 3
1	α	β	γ
2	γ	α	β
3	β	γ	α

- Four 3-level factors
  - column: 1,2,3
  - row: 1,2,3
  - Latin: A,B,C
  - Greek: α,β,γ

d.f. of residuals in LSD (LNp.30) = $(k-1)(k-2) \geq k-1$ if $k \geq 3 \Rightarrow$ may add one more factor

- In the DM, for 1st LS
  column ⊥ row
  column ⊥ Latin
  row ⊥ Latin
- In the DM, for 2nd LS
  column ⊥ row
  column ⊥ Greek
  row ⊥ Greek
- Then, we would like to have Latin ⊥ Greek
  how?

★ Design matrix → an OA of strength 2 with 4 factors

column	row	Latin	Greek
1	1	A	α
1	2	B	γ
1	3	C	β
2	1	B	β
2	2	C	α
2	3	A	γ
3	1	C	γ
3	2	A	β
3	3	B	α

?

(column, row, Latin) → an OA of strength 2 with 3 factors

(column, row, Greek) → an OA of strength 2 with 3 factors

Graeco-Latin Square Design (cont.)

- Graeco-Latin square design is useful for studying four factors allowing one more factor to be studied than in LS under main-effect-only models.
- The four factors can be
  - 1 treatment and 3 blocking factors, or
  - 2 treatment and 2 blocking factors, etc.

  ⋮

Q: What are the structure in their EUs? (check LNp.27) k^2 EUs

$\boxed{\text{total d.f.} = k^2 \Rightarrow \text{not enough d.f. for interactions}}$

★ conceptual model: ← before exp't

$(k-1)$ parameters

$y \sim \beta_0 + \text{row} + \text{column} + \text{Latin} + \text{Greek} + \varepsilon$ ← [main-effect-only model ⇒ no interactions (Why?)]

sum codings → W_r W_c W_L W_G ⇒ $W_0 \perp W_r \perp W_c \perp W_L \perp W_G \perp \Omega^\perp$

In the DM, level combinations of any 2 factors appear equally often.
⇒ the DM is an OA of strength 2 (higher strength, better orthogonality)
⇒ guarantee main-effect spaces (under sum codings) are mutually orthogonal.

Model and ANOVA in Graeco-Latin Square Design

- Model: ← over-parameterized

$$y_{ijlm} = \eta + \alpha_i + \beta_j + \tau_l + \zeta_m + \varepsilon_{ijlm}, \quad i,j,l,m = 1,\ldots,k.$$

(k) (k) (k) (k) ← l, m are functions of (i, j)

- Similar interpretation as in LSD, and ζ_m is the mth effect of Greek letters).
- F-test and Tukey's multiple comparisons similar formulae.

Table 19: ANOVA Table for Graeco-Latin Square Design ← use (P4) in LNp.2-36

	Source	Degrees of Freedom	Sum of Squares	MS =?	E(MS) =?
W_r	row	$k-1$	$k\sum_{i=1}^{k}(\bar{y}_{i\cdots}-\bar{y}_{\cdots\cdot})^2$		
W_c	column	$k-1$	$k\sum_{j=1}^{k}(\bar{y}_{\cdot j\cdot\cdot}-\bar{y}_{\cdots\cdot})^2$		
W_L	Latin letter	$k-1$	$k\sum_{l=1}^{k}(\bar{y}_{\cdot\cdot l\cdot}-\bar{y}_{\cdots\cdot})^2$		
W_G	Greek letter	$k-1$	$k\sum_{m=1}^{k}(\bar{y}_{\cdots m}-\bar{y}_{\cdots\cdot})^2$		
$\Omega^\perp$	residual	$(k-3)(k-1)$	by subtraction	□ ← $\hat{\sigma}^2$	
$\mathbb{R}^N \ominus W_0$	total	k^2-1	$\sum_{(i,j,l,m)\in S}(y_{ijlm}-\bar{y}_{\cdots\cdot})^2$		

$\Omega \ominus W_0$ = $W_r \oplus W_c \oplus W_L \oplus W_G$

Sum of Squares = $\|P_{W_i}(\underset{\sim}{y})\|^2$

1 ← β_0

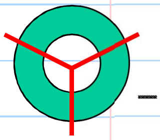
$k^2-1-4(k-1) = (k-1)(k+1-4)$
$= (k-1)(k-2) - (k-1)$ ← d.f. of residuals in LSD

❖ **Reading**: textbook, 3.7

Incomplete Blocking

★ response: wear amount
★ treatment factor component (qualitative) 4 levels – A, B, C, D
★ block factor tire (qualitative) 4 levels – 1, 2, 3, 4 block size = 3
★ Exp'tal units: a section (1/3) of a tire, 12 EUs

- Example: Tire wear experiment.
  - Compare four components A, B, C, D in terms of wear. (t)
  - Because of manufacturing limitations, each of 4 tires can be divided into only three sections with each section being made of one compound. (b, k)
  - The 3 sections of a tire are subjected to same road conditions.



restricted randomization

tire 1 (block 1)			•••	tire 4 (block 4)		
EU11	EU12	EU13	•••	EU41	EU42	EU43
A, B, C (r)				B, C, D (r)		

⇒ within-block comparison is not enough

★ Design matrix

block	Tire	Component	y	treatment
	1	A	238	
	1	B		
	1	C		
	2	A	•	
	2	B		
	2	D	•	
	3	A		
	3	C	•	
	3	D		
	4	B		
	4	C		
	4	D	412	

not an OA ⇒ lose orthogonality $\Omega_b \times \Omega_t$

Data matrix

before exp't → ★ conceptual model (same as RBD):

$y \sim \beta_0 + \text{block} + \text{treatment} + \varepsilon$ or $y_{ij} = \eta + \alpha_i + \tau_j + \varepsilon_{ij}$ under sum coding

(b-1) parameters (block), (t-1) parameters (treatment)

$\Omega_b = \text{span}\{X_b\}$
$\Omega_t = \text{span}\{X_t\}$

- Blocking is *incomplete* if the number of treatments t is greater than the block size k (i.e., $t > k$). This happens if the nature of blocking makes it difficult to form blocks of large size. $\boxed{k = t \text{ or } k = t \times \ell}$

Note: $W_t \neq \Omega_t$ (check LNp.2-32)

- On the other hand, RBD (LNp.4-8) has *complete* blocking.

Table 20: Wear Data, Tire Experiment

$t=4$, $b=4$, $k=3$

blocks: Tire	A	B	C	D	
1	238 1	238 1	279 1	0	$\bar{y}_{1.}$
2	196 1	213 1	0	308 1	$\bar{y}_{2.}$
3	254 1	0	334 1	367 1	$\bar{y}_{3.}$
4	0	312 1	421 1	412 1	$\bar{y}_{4.}$
$-\bar{y}_{..}$	$\bar{y}_{.1}$	$\bar{y}_{.2}$	$\bar{y}_{.3}$	$\bar{y}_{.4}$	$\bar{y}_{..}$
sum codings	τ_1	τ_2	τ_3	$(\tau_4 = -\tau_1 - \tau_2 - \tau_3)$	

one-way ← $y \sim \beta_0 + \text{treatment} + \varepsilon \cdots (\triangle)$

$y \sim \beta_0 + \text{block} + \text{treatment} + \varepsilon \cdots (*)$

$H_1 X_b \beta_b$ $\qquad X = [\mathbb{1}\ X_b\ X_t]$

A vs. B

$\bar{y}_{.1} - \bar{y}_{.2}$ ← 2 WBC, 1 BBC ← biased by block effect

- within block comparison (WBC): block effect can be eliminated
- between block comparison (BBC): block effect remains

★ 2 WBC, 1 BBC for A vs. C, A vs. D, B vs. C, B vs. D, C vs. D (λ) ← balanced incomplete block design

$E(\bar{y}_{.1} - \bar{y}_{..}) = \frac{1}{3}(3\eta + \alpha_1 + \alpha_2 + \alpha_3 + 3\tau_1)$
$\quad - \frac{1}{12}(12\eta + 3\alpha_1 + 3\alpha_2 + 3\alpha_3 + 3\alpha_4 + 3\tau_1 + 3\tau_2 + 3\tau_3 + 3\tau_4)$
$= (\tau_1 - \bar{\tau}) - \frac{1}{3}(\alpha_4 - \bar{\alpha}) \neq \tau_1 - \bar{\tau}$ ← $=0$

Let $S_{.j} = \frac{bk}{t}\bar{y}_{.j}$ and $Q_j = S_{.j} - \sum_i n_{ij}\bar{y}_{i.}$ (n_{ij}: # of treatment j in block i)

$E(Q_1) = E(S_{.1} - \bar{y}_{1.} - \bar{y}_{2.} - \bar{y}_{3.})$
$= (3\eta + 3\tau_1 + \alpha_1 + \alpha_2 + \alpha_3) - \frac{1}{3}(3\eta + 3\alpha_1 + \tau_1 + \tau_2 + \tau_3)$
$\quad - \frac{1}{3}(3\eta + 3\alpha_2 + \tau_1 + \tau_2 + \tau_4) - \frac{1}{3}(3\eta + 3\alpha_3 + \tau_1 + \tau_3 + \tau_4)$
$= (8/3)(\tau_1 - \bar{\tau}) \Leftarrow (\lambda t/k)(\tau_1 - \bar{\tau})$

$\Rightarrow E[(k/\lambda t)Q_j] = \tau_j - \bar{\tau} \Rightarrow \hat{\tau}_j = (k/\lambda t)Q_j$

(exercise) $\boxed{\text{cov}(\hat{\beta}) = (X^TX)^{-1}\sigma^2}$ $\quad \boxed{\hat{\beta} = (X^TX)^{-1}X^TY \text{ under } (*)}$

$$\text{cov}\left(\begin{bmatrix}\hat{\tau}_1\\ \vdots \\ \hat{\tau}_{t-1}\end{bmatrix}\right) = \begin{bmatrix} v & & u \\ & \ddots & \\ u & & v\end{bmatrix}\sigma^2, \text{ where } u = \frac{-k}{\lambda t^2},\ v = \frac{k(t-1)}{\lambda t^2}$$

$\Rightarrow \text{var}(\hat{\tau}_i - \hat{\tau}_j) = (2v - 2u)\sigma^2 = (2k/\lambda t)\sigma^2$

estimated from residuals of (*) with d.f. $bk - b - t + 1$

check LM, LNp.3-12~13