Graeco-Latin Square Design

level combinations of Latin & Greek

- Two Latin squares are *orthogonal* if each pair of letters (Latin and Greek) appears once in the two squares, when superimposed. The super-imposed square is called a *Graeco-Latin square*.

or appears equally often

row	column 1	column 2	column 3
1	Aα	Bβ	Cγ
2	Bγ	Cα	Aβ
3	Cβ	Aγ	Bα

←

orthogonal

row	column 1	column 2	column 3
1	A	B	C
2	B	C	A
3	C	A	B

1st LS

$+$

row	column 1	column 2	column 3
1	α	β	γ
2	γ	α	β
3	β	γ	α

2nd LS

- Four 3-level factors
  - column : 1, 2, 3
  - row : 1, 2, 3
  - Latin : A, B, C
  - Greek : α, β, γ

d.f. of residuals in LSD (LNp.30) $= (k-1)(k-2) \geq k-1$ if $k \geq 3 \Rightarrow$ may add one more factor

- In the DM, for 1st LS
  - column ⊥ row
  - column ⊥ Latin
  - row ⊥ Latin
- In the DM, for 2nd LS
  - column ⊥ row
  - column ⊥ Greek
  - row ⊥ Greek
- Then, we would like to have Latin ⊥ Greek
  how?

★ Design matrix → an OA of strength 2 with 4 factors

column	row	Latin	Greek
1	1	A	α
1	2	B	γ
1	3	C	β
2	1	B	β
2	2	C	α
2	3	A	γ
3	1	C	γ
3	2	A	β
3	3	B	α

?

→ an OA of strength 2 with 3 factors

→ an OA of strength 2 with 3 factors

Graeco-Latin Square Design (cont.)

- Graeco-Latin square design is useful for studying four factors allowing one more factor to be studied than in LS under main-effect-only models.

- The four factors can be
  - 1 treatment and 3 blocking factors, or
  - 2 treatment and 2 blocking factors, etc.

  ⋮

Q: What are the structure in their EUs? (check LNp.27) ← k^2 EUs

total d.f. $= k^2 \Rightarrow$ not enough d.f. for interactions →

★ conceptual model: ← before exp't

$(k-1)$ parameters

$y \sim \beta_0 + \text{row} + \text{column} + \text{Latin} + \text{Greek} + \varepsilon$ ← main-effect-only model ⇒ no interactions (Why?)

sum codings → W_r W_c W_L W_G $\Rightarrow W_0 \perp W_r \perp W_c \perp W_L \perp W_G \perp \Omega^\perp$

In the DM, level combinations of any 2 factors appear equally often
⇒ the DM is an OA of strength 2 (higher strength, better orthogonality)
⇒ guarantee main-effect spaces (under sum codings) are mutually orthogonal.

Model and ANOVA in Graeco-Latin Square Design

- Model: ← over-parameterized

$$y_{ijlm} = \eta + \alpha_i + \beta_j + \tau_l + \zeta_m + \varepsilon_{ijlm}, \quad i,j,l,m = 1,\ldots,k. \leftarrow$$

$\underset{(k)}{\alpha_i}$ $\underset{(k)}{\beta_j}$ $\underset{(k)}{\tau_l}$ $\underset{(k)}{\zeta_m}$

l, m are functions of (i, j)

  - Similar interpretation as in LSD, and ζ_m is the mth effect of Greek letters).
  - F-test and Tukey's multiple comparisons similar formulae.

Table 19: ANOVA Table for Graeco-Latin Square Design — use (P4) in LNp.2-36

	Source	Degrees of Freedom	Sum of Squares	MS	E(MS)
$\Omega \ominus W_0$				=?	=?
W_r	row	$k-1$	$k\sum_{i=1}^{k}(\bar{y}_{i\cdots}-\bar{y}_{\cdots\cdot})^2$ ← $\|P_{W_i}(y)\|^2$		
W_c	column	$k-1$	$k\sum_{j=1}^{k}(\bar{y}_{\cdot j\cdot\cdot}-\bar{y}_{\cdots\cdot})^2$		
W_L	Latin letter	$k-1$	$k\sum_{l=1}^{k}(\bar{y}_{\cdot\cdot l\cdot}-\bar{y}_{\cdots\cdot})^2$		
W_G	Greek letter	$k-1$	$k\sum_{m=1}^{k}(\bar{y}_{\cdots m}-\bar{y}_{\cdots\cdot})^2$		
$\Omega^\perp$	residual	$(k-3)(k-1)$	by subtraction	$\square \leftarrow \hat{\sigma}^2$	
$\mathbb{R}^N \ominus W_0$	total	k^2-1	$\sum_{(i,j,l,m)\in S}(y_{ijlm}-\bar{y}_{\cdots\cdot})^2$		

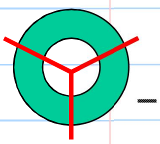
$\Omega \ominus W_0 = W_r \oplus W_c \oplus W_L \oplus W_G$; B_0

$k^2-1-4(k-1) = (k-1)(k+1-4)$
$= (k-1)(k-2) - (k-1)$
↑ d.f. of residuals in LSD

❖ **Reading**: textbook, 3.7

Incomplete Blocking

- Example: Tire wear experiment.
  - Compare four components A, B, C, D in terms of wear. (t)
  - Because of manufacturing limitations, each of 4 tires (b) can be divided into only three (k) sections with each section being made of one compound.
  - The 3 sections of a tire are subjected to same road conditions.



✱ response: wear amount
✱ treatment factor
component (qualitative)
4 levels – A, B, C, D

✱ block factor
tire (qualitative)
4 levels – 1, 2, 3, 4
block size = 3

✱ Exp'tal units: a section (1/3) of a tire, 12 EUs

tire 1 (block 1): EU11 EU12 EU13 ••• tire4 (block 4): EU41 EU42 EU43

restricted randomization: A, B, C ... B, C, D

⇒ within-block comparison is not enough

✱ Design matrix

block	Tire	Component	y	treatment
	1	A	238	
	1	B	•	
	1	C		
	2	A	•	
	2	B		
	2	D		
	3	A	•	
	3	C		
	3	D		
	4	B		
	4	C		
	4	D	412	

Data matrix

not an OA ⇒ lose orthogonality $\Omega_b \cancel{\perp} \Omega_t$

before exp't → ✱ conceptual model (same as RBD):

$y \sim B_0 + \text{block} + \text{treatment} + \varepsilon$ or $y_{ij} = \eta + \alpha_i + \tau_j + \varepsilon_{ij}$ under sum coding

(b−1) parameters, (t−1) parameters

- Blocking is *incomplete* if the number of treatments t is greater than the block size k (i.e., $t > k$). This happens if the nature of blocking makes it difficult to form blocks of large size. $k = t$ or $k = t \times l$

$\Omega_b = \text{span}\{X_b\}$
$\Omega_t = \text{span}\{X_t\}$

Note: $W_t \neq \Omega_t$ (check LNp.2-32)

- On the other hand, RBD (LNp.4-8) has *complete* blocking.

Table 20: Wear Data, Tire Experiment

$t=4$, $b=4$, $k=3$

Tire	Compound A	B	C	D	
1	238	238	279		$\bar{y}_{1\cdot}$
2	196	213		308	$\bar{y}_{2\cdot}$
3	254		334	367	$\bar{y}_{3\cdot}$
4		312	421	412	$\bar{y}_{4\cdot}$
$-\bar{y}_{\cdot\cdot}$	$\bar{y}_{\cdot 1}$	$\bar{y}_{\cdot 2}$	$\bar{y}_{\cdot 3}$	$\bar{y}_{\cdot 4}$	$\bar{y}_{\cdot\cdot}$

blocks

sum codings $\to$ τ_1, τ_2, τ_3 ($\tau_4 = -\tau_1-\tau_2-\tau_3$)

one-way $\leftarrow$ $y \sim \beta_0 + \text{treatment} + \varepsilon \cdots (\triangle)$

$y \sim \beta_0 + \text{block} + \text{treatment} + \varepsilon \cdots (*)$

$H_1 X_b B_b$ $\quad X = [\mathbb{1}\ X_b\ X_t]$

$\boxed{\text{A vs. B}}$

$\bar{y}_{\cdot 1} - \bar{y}_{\cdot 2} \leftarrow$ 2 WBC, 1 BBC $\leftarrow$ biased by block effect

- within block comparison (WBC): block effect can be eliminated
- between block comparison (BBC): block effect remains

λ

☆ 2 WBC, 1 BBC for A vs. C, A vs. D, B vs. C, B vs. D, C vs. D $\Leftarrow$ balanced incomplete block design

$E(\bar{y}_{\cdot 1} - \bar{y}_{\cdot\cdot}) = \frac{1}{3}(3\eta + \alpha_1 + \alpha_2 + \alpha_3 + 3\tau_1)$
$\quad - \frac{1}{12}(12\eta + 3\alpha_1 + 3\alpha_2 + 3\alpha_3 + 3\alpha_4 + 3\tau_1 + 3\tau_2 + 3\tau_3 + 3\tau_4)$
$= (\tau_1 - \bar{\tau}) - \frac{1}{3}(\alpha_4 - \bar{\alpha}) \neq \tau_1 - \bar{\tau}$ $\quad \leftarrow = 0$

Let $S_{\cdot j} = \frac{bk}{t}\bar{y}_{\cdot j}$ and $Q_j = S_{\cdot j} - \sum_i n_{ij}\bar{y}_{i\cdot}$

$E(Q_1) = E(S_{\cdot 1} - \bar{y}_{1\cdot} - \bar{y}_{2\cdot} - \bar{y}_{3\cdot})$
$= (3\eta + 3\tau_1 + \alpha_1 + \alpha_2 + \alpha_3) - \frac{1}{3}(3\eta + 3\alpha_1 + \tau_1 + \tau_2 + \tau_3)$
$\quad - \frac{1}{3}(3\eta + 3\alpha_2 + \tau_1 + \tau_2 + \tau_4) - \frac{1}{3}(3\eta + 3\alpha_3 + \tau_1 + \tau_3 + \tau_4)$
$= (8/3)(\tau_1 - \bar{\tau}) \Leftarrow (\lambda t/k)(\tau_1 - \bar{\tau})$

$\Rightarrow E[(k/\lambda t)Q_j] = \tau_j - \bar{\tau} \Rightarrow \hat{\tau}_j = (k/\lambda t)Q_j$

(exercise) $\boxed{\text{cov}(\hat{\beta}) = (X^TX)^{-1}\sigma^2}$ $\quad \boxed{\hat{\beta} = (X^TX)^{-1}X^TY \text{ under } (*)}$

$$\text{cov}\left(\begin{bmatrix}\hat{\tau}_1\\ \vdots \\ \hat{\tau}_t\end{bmatrix}\right) = \begin{bmatrix} v & & u \\ & \ddots & \\ u & & v\end{bmatrix}\sigma^2, \text{ where } u = \frac{k}{\lambda t^2},\ v = \frac{k(t-1)}{\lambda t^2}$$

$\Rightarrow \text{var}(\hat{\tau}_i - \hat{\tau}_j) = (2v - 2u)\sigma^2 = (2k/\lambda t)\sigma^2$

estimated from residuals of $(*)$ with d.f. $bk-b-t+1$

check LM, LN p. 3-12~13