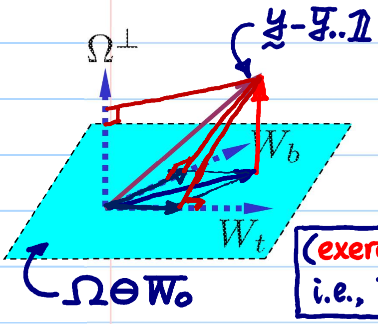
ANOVA

$\|\underset{\sim}{y}\|^2 = \|P_{W_0}(\underset{\sim}{y})\|^2 + \|P_{W_b}(\underset{\sim}{y})\|^2 + \|P_{W_t}(\underset{\sim}{y})\|^2 + \|P_{\Omega^\perp}(\underset{\sim}{y})\|^2$

- **ANOVA decomposition**. Subtracting $\bar{y}_{..}$, squaring both sides and summing over i and j yields

remove $P_{W_0}(\underset{\sim}{y})$



$$\underbrace{\sum_{i=1}^{b}\sum_{j=1}^{k}(y_{ij}-\bar{y}_{..})^2}_{\text{total variation in } \underset{\sim}{y}} = \underbrace{\sum_{i=1}^{b}k(\bar{y}_{i\cdot}-\bar{y}_{..})^2}_{\|P_{W_b}(\underset{\sim}{y})\|^2} + \underbrace{\sum_{j=1}^{k}b(\bar{y}_{\cdot j}-\bar{y}_{..})^2}_{\|P_{W_t}(\underset{\sim}{y})\|^2} + \underbrace{\sum_{i=1}^{b}\sum_{j=1}^{k}(y_{ij}-\bar{y}_{i\cdot}-\bar{y}_{\cdot j}+\bar{y}_{..})^2}_{\|P_{\Omega^\perp}(\underset{\sim}{y})\|^2}$$

$$= SS_b + SS_t + SS_r.$$

total variation in $\underset{\sim}{y}$ = $\|P_{\mathbb{R}^N \ominus W_0}(\underset{\sim}{y})\|^2$ ~ $W_b \oplus W_t \oplus \Omega^\perp$

(exercise) Note ∵ orthogonality, i.e., $W_0 \perp W_b \perp W_t \perp \Omega^\perp$

sources of variation

Note. In this case, ∵ orthogonality, anova($y \sim B_0$ + block + treatment) & anova($y \sim B_0$ + treatment + block) have same output

Table 5: ANOVA Table for Randomized Block Design

usually is the 1st effects to enter the model

Source	Degrees of Freedom	Sum of Squares	MS =?	E_Ω(MS) =?
block	$b-1$	$\sum_{i=1}^{b}k(\bar{y}_{i\cdot}-\bar{y}_{..})^2$	(exercise) check (P4) in LNp.2-36 & LNp.3-6	
treatment	$k-1$	$\sum_{j=1}^{k}b(\bar{y}_{\cdot j}-\bar{y}_{..})^2$		
residual	$(b-1)(k-1)$	$\sum_{i=1}^{b}\sum_{j=1}^{k}(y_{ij}-\bar{y}_{i\cdot}-\bar{y}_{\cdot j}+\bar{y}_{..})^2$		
total	$bk-1$	$\sum_{i=1}^{b}\sum_{j=1}^{k}(y_{ij}-\bar{y}_{..})^2$		

same as the df's for interactions

can do $k \times b$-run RBD because of assuming main-effect model in conceptual modeling

Testing and Multiple Comparisons

ANOVA & multiple comparison for RBD use the same principles as in one-way layout, except that the variation caused by block factor must be removed from error variation. how? adding block effects into model

- $H_0 : \tau_1 = \cdots = \tau_k$, can be tested by using the F-statistic:

$$F = \frac{SS_t/(k-1)}{SS_r/(b-1)(k-1)}, \tag{3}$$

$RSS_\omega - RSS_\Omega$ → SS_t; $df_\omega - df_\Omega$ → $(k-1)$; RSS_Ω → SS_r; df_Ω → $(b-1)(k-1)$

$\Omega: y \sim B_0 + \text{Girder} + \text{Method} + \varepsilon$ $(y = \eta + \alpha_i + \tau_j + \varepsilon)$

$\omega: y \sim B_0 + \text{Girder} + \varepsilon$ $(y = \eta + \alpha_i + \varepsilon)$

Recall. LNp3-9~13

The F test rejects H_0 at level α if $F > F_{k-1,(b-1)(k-1),\alpha}$.

- *If H_0 is rejected, multiple comparisons of the τ_j should be performed.*

  – The t-statistics for making multiple comparisons :

$\text{Var}(\bar{y}_{\cdot j} - \bar{y}_{\cdot i}) = \sigma^2/b + \sigma^2/b$

$H_0^{ij}: \tau_i = \tau_j$ $(\tau_j - \tau_i = 0)$

– for specific (i,j) → use t-dist. for null.

– for all (i,j)'s

$$t_{ij} = \frac{\bar{y}_{\cdot j} - \bar{y}_{\cdot i} - 0}{\hat{\sigma}\sqrt{1/b + 1/b}}, \tag{4}$$

$\hat{\tau}_j - \hat{\tau}_i$

block effects removed

where $\hat{\sigma}^2$ is the mean square error in the ANOVA table. $\hat{\sigma}^2 = \frac{SS_r}{(b-1)(k-1)}$

– At level α, the Tukey multiple comparison method identifies "treatments i and j as different" if

$$|t_{ij}| > \frac{1}{\sqrt{2}} q_{k,(b-1)(k-1),\alpha}.$$

studentized range dist.

cf. Alternative: Bonferroni Method

Simultaneous Confidence Intervals

Recall. duality between confidence interval and test.

model: $y = \eta + \text{block} + \text{treatment} + \varepsilon$

τ_j's

$H_0^{ij}: \tau_j - \tau_i = C$ for all (i, j)

- By solving

$$\frac{|(\bar{y}_{\cdot j} - \bar{y}_{\cdot i}) - (\tau_j - \tau_i)|}{\hat{\sigma}\sqrt{2/b}} \leq \frac{1}{\sqrt{2}} q_{k,(b-1)(k-1),\alpha}$$

C

of τ_j's

for $\tau_j - \tau_i$, the simultaneous confidence intervals for $\tau_j - \tau_i$ are

$$\bar{y}_{\cdot j} - \bar{y}_{\cdot i} \pm q_{k,(b-1)(k-1),\alpha} \frac{\hat{\sigma}}{\sqrt{b}}$$

$\hat{\sigma}$ ← block effects removed

for all i and j pairs.

Analysis of Girder Experiment : F-test

∵ orthogonality ∴ same as

$\Omega_b: y \sim \beta_0 + \text{girder} + \text{method} + \varepsilon$
$\omega_b: y \sim \beta_0 + \text{method} + \varepsilon$

Table 6: ANOVA Table, Girder Experiment

anova ($y \sim \beta_0 + \text{girder} + \text{method}$)

$\Omega_b: y \sim \beta_0 + \text{girder} + \varepsilon$
$\omega_b: y \sim \beta_0 + \varepsilon$

Source	Degrees of Freedom	Sum of Squares	Mean Squares	F
girder (not of interest)	8 = 9−1 (6−1)	0.089 ← $\|P_{W_b}(y)\|^2$	0.011	1.62
method (of main interest)	3 = 4−1 (k−1)	1.514 ← $\|P_{W_t}(y)\|^2$	0.505	73.03
residual	24 = (b−1)(k−1)	0.166 ← $\|P_{\Omega^\perp}(y)\|^2$	0.007 ← $\hat{\sigma}^2$	
total	35 ← 36 obs	← $\|P_{\mathbb{R}^N \ominus W_0}(y)\|^2 = \|y\|^2 - \|P_{W_0}(y)\|^2$		

1.62 ← not significant ⇒ homogeneous btwn blocks ⇒ (maybe) not necessary to consider it as a block factor in future exp't.

- The F-statistic in (3) (LNp.4-11) has the value

$$\frac{1.514/3}{0.166/24} = 73.03.$$

$\Omega_t: y \sim \beta_0 + \text{girder} + \text{method} + \varepsilon$
$\omega_t: y \sim \beta_0 + \text{girder} + \varepsilon$

∵ orthogonality ∴ same as
$\Omega_t: y \sim \beta_0 + \text{method} + \varepsilon$
$\omega_t: y \sim \beta_0 + \varepsilon$

Therefore, the p-value for testing the difference between methods is $Prob(F_{3,24} > 73.03)$=0.00. The small p-value suggests that the methods are different.

Analysis of Girder Experiment : Multiple Comparisons

Answer to "how different" problem

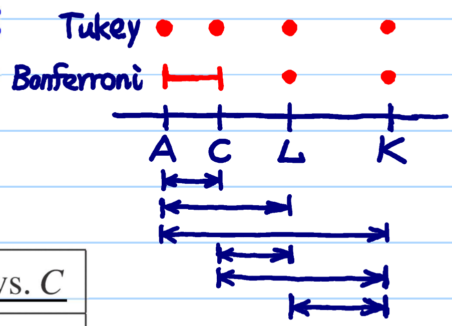


Table 7: Multiple Comparison t Statistics, Girder Experiment

	A vs. K	A vs. L	A vs. C	K vs. L	K vs. C	L vs. C
t_{ij}	13.91	6.92	2.82	−6.99	−11.09	−4.10

- The means for the four methods, A for Aarau, K for Karlsruhe, L for Lehigh and C for Cardiff are 0.7949, 1.3401, 1.0662 and 0.9056.
  $\bar{y}_{\cdot j}$'s
- The multiple comparison t-statistics based on (4) (LNp.4-11) are displayed in Table 7. For example, the A vs. K t-statistic is

$$t_{12} = \frac{1.3401 - 0.7949}{\underbrace{\sqrt{0.007}}_{\hat\sigma^2}\underbrace{\sqrt{2/9}}_{2(1/b)}} = 13.91.$$

\# of tests

With $\alpha = 0.05$, $t_{24, 0.05/(6\times 2)} = 2.875$ for the Bonferroni method. Since $k = 4$ and $\binom{k}{2} = 6$, $\frac{1}{\sqrt{2}} q_{4,24,0.05} = \frac{3.90}{1.414} = 2.758$ for the Tukey method. Again, Tukey method is more powerful. (Why ?)

\# of $\bar{y}_{\cdot j}$'s

❖ **Reading**: textbook, 3.2

Two-way layout

treatment factors (crossing)

cf. nesting e.g. plant < A → operator − 1, 2, 3; B → operator − 1′, 2′, 3′

- **Bolt experiment** : The goals was to test if there is any difference between two test media (bolt, mandrel) and among three plating methods (C&W, HT, P&O). Response y is the torque of the locknut.

★ response : torque
★ treatment factors : ① media (qualitative) ② plating (qualitative)
2 levels − B, M 3 levels − C, H, P
★ Exp'tal units : a locknut, 60 EUs
homogeneous → EU1 • • • EU60
completely randomized → (media, plating) level combinations ⇒ 6 treatments

Table 8: Torque Data, Bolt Experiment

	C&W	HT	P&O
Bolt	20, 16, 17, 18, 15, 16, 19, 14, 15, 24	26, 40, 28, 38, 38, 30, 26, 38, 45, 38	25, 40, 30, 17, 16, 45, 49, 33, 30, 20
Mandrel	24, 18, 17, 17, 15, 23, 14, 18, 12, 11	32, 22, 30, 35, 32, 28, 27, 28, 30, 30	10, 13, 17, 16, 15, 14, 11, 14, 15, 16

★ Each treatment repeats 10 times ⇒ 10 replicates (Why need replicate?)
★ Design matrix Data matrix

$$\begin{array}{ccc} \text{media} & \text{plating} & y \\ \left[\begin{array}{c} B \\ B \\ B \\ M \\ M \\ M \end{array}\right. & \begin{array}{c} C \\ H \\ P \\ C \\ H \\ P \end{array} & \left.\begin{array}{c} \square\cdots\square \\ \vdots \\ \vdots \\ \square\cdots\square \end{array}\right] \end{array}$$

total dfs: 60
− 6 for β
− 54 for σ^2

different EU structure
different randomization scheme

- This is similar to RBD. The only difference is that we have two treatment factors instead of one treatment factor and one block factor. Also interested in assessing interaction effects between the two treatment factors. In blocking, block×treatment interaction is assumed negligible.

Conceptual model : ← before exp't

$$y \sim \beta_0 + \underbrace{\text{media}}_{1 \text{ parameter}} + \underbrace{\text{plating}}_{2 \text{ parameters}} + \underbrace{\text{media}\times\text{plating}}_{1\times 2 = 2 \text{ parameters}} + \varepsilon$$

Model

$y \sim \beta_0 + \underset{①\ I-1}{\text{media}} + \underset{②\ J-1}{\text{plating}} + \underset{②\ (I-1)(J-1)}{\text{media}\times\text{plating}} + \varepsilon$

- Model : μ_{ij}

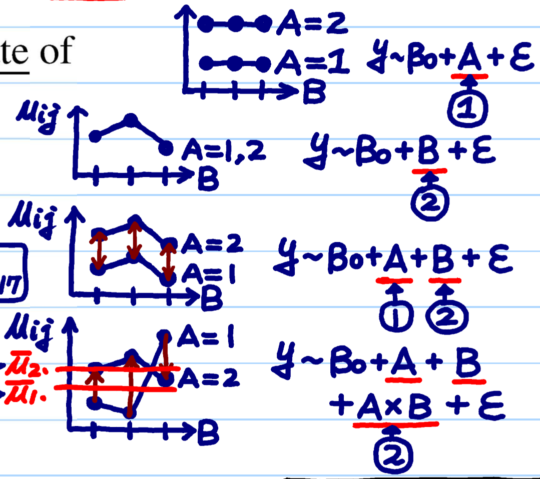
$\boxed{\text{functional form}} \rightarrow$
$$y_{ijl} = \underbrace{\eta + \alpha_i + \beta_j + \omega_{ij}}_{\mu_{ij}} + \varepsilon_{ijl}, \quad (5)$$
① ②I ③J ⑥I×J $\leftarrow$ $\boxed{\text{over-parameterized}} \rightarrow \boxed{\text{OK for the tests of ANOVA}}$

$i = 1, \ldots, I$; $j = 1, \ldots, J$; $l = 1, \ldots, n$, where $E(y_{ijl}) = \mu_{ij}$ $\boxed{\text{LM, LNp.8-17~18}}$

$\boxed{\text{Q: What do } \alpha_i\text{'s \& } \beta_j\text{'s mean if model contain } \omega_{ij}\text{'s?}}$

y_{ijl}	=	observation for the lth replicate of the ith level of factor A and the jth level of factor B,
α_i	=	ith main effect for A,
β_j	=	jth main effect for B,
ω_{ij}	=	(i,j)th interaction effect between A and B, and
ε_{ijl}	=	errors, independent $N(0,\sigma^2)$.

(exercise)
$\underline{\mu}_{i\cdot} = \eta\mathbb{1} + X_\alpha \alpha$
$\underline{\mu}_{\cdot j} = \eta\mathbb{1} + X_\beta \beta$
(check LNp.4-17)

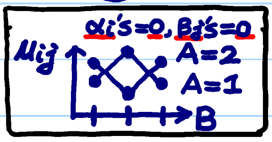


Under sum codings,

$$
\begin{array}{c|cccc|c}
 & & & B\ (j) & & \\
\hline
 & \omega_{11} & \omega_{12} & \cdots & \omega_{1J} & \Sigma=0 \\
A & \omega_{21} & \omega_{22} & \cdots & \omega_{2J} & \Sigma=0 \\
(i) & \vdots & \vdots & \ddots & \vdots & \\
 & \omega_{I1} & \omega_{I2} & \cdots & \omega_{IJ} & \Sigma=0 \\
\hline
 & \Sigma=0 & & & &
\end{array}
$$

$\alpha_I = -(\alpha_1 + \cdots + \alpha_{I-1}) \Rightarrow I \rightarrow I-1$

$\beta_J = -(\beta_1 + \cdots + \beta_{J-1}) \Rightarrow J \rightarrow J-1$



$\boxed{\#\text{ of constraints} = I+J-1} \leftarrow$
$\omega_{iJ} = -(\omega_{i1} + \cdots + \omega_{iJ-1}),\ i = 1, \ldots, I$
$\omega_{Ij} = -(\omega_{1j} + \cdots + \omega_{I-1\,j}),\ j = 1, \ldots, J$
$\cancel{\sum_{i=1}^{I}\sum_{j=1}^{J}\omega_{ij} = 0}$

$\Rightarrow IJ \rightarrow IJ-(I+J-1) = (I-1)(J-1)$

Estimation

$\boxed{\text{matrix form}: Y = X\gamma + \varepsilon}$ $\boxed{\text{sum codings}}$

$\boxed{\text{model matrix } X}$

$\boxed{\text{design matrix}}$

$\boxed{x_{ij} = x_i \times x_j}$

$$
\begin{array}{cc|c|cc|ccc|ccc|ccc}
 & & & & -\alpha_1 & & -\beta_1-\beta_2 & & & -\omega_{11}-\omega_{12} & & -\omega_{11} & -\omega_{12} & -\omega_{21}-\omega_{22} \\
 & & & & \| & & \| & & & \| & & \| & \| & \| \\
(i,j) & \mu_{ij} & \eta & \alpha_1 & \alpha_2 & \beta_1 & \beta_2 & \beta_3 & \omega_{11} & \omega_{12} & \omega_{13} & \omega_{21} & \omega_{22} & \omega_{23} \\
1\,1 & \mu_{11} & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1\,2 & \mu_{12} & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1\,3 & \mu_{13} & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
2\,1 & \mu_{21} & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
2\,2 & \mu_{22} & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
2\,3 & \mu_{23} & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}
$$

Under sum codings (red columns):

$$
\begin{array}{c|c|c|cc|cc}
 & \eta & \alpha_1 & \beta_1 & \beta_2 & \omega_{11} & \omega_{12} \\
\mu_{11} & 1 & 1 & 1 & 0 & 1 & 0 \\
\mu_{12} & 1 & 1 & 0 & 1 & 0 & 1 \\
\mu_{13} & 1 & 1 & -1 & -1 & -1 & -1 \\
\mu_{21} & 1 & -1 & 1 & 0 & -1 & 0 \\
\mu_{22} & 1 & -1 & 0 & 1 & 0 & -1 \\
\mu_{23} & 1 & -1 & -1 & -1 & 1 & 1
\end{array}
$$

span $\downarrow W_0$; $\downarrow$ span $W_\alpha(X_\alpha)$; $\downarrow$ span $W_\beta(X_\beta)$; $\downarrow$ span $W_\omega(X_\omega)$ $\leftarrow \{\mathbb{1}\} \perp \{x_i\} \perp \{x_j\} \perp \{x_{ij}\} \leftarrow \boxed{\because \text{balance}}$

$\boxed{x_i \perp x_j \perp x_{ij}}$
x_i's: not orthogonal
x_j's: not orthogonal
x_{ij}'s: not orthogonal

$$\hat{\gamma} = (X^TX)^{-1}X^TY = \begin{bmatrix} y_{\cdots} \\ (X_\alpha^TX_\alpha)^{-1}X_\alpha^TY \\ (X_\beta^TX_\beta)^{-1}X_\beta^TY \\ (X_\omega^TX_\omega)^{-1}X_\omega^TY \end{bmatrix} \begin{matrix} \rightarrow \hat{\eta} \\ \rightarrow \hat{\alpha}_i\text{'s} \\ \rightarrow \hat{\beta}_j\text{'s} \\ \rightarrow \hat{\omega}_{ij}\text{'s} \end{matrix}$$ (exercise)

$X\hat{\gamma} = X(X^TX)^{-1}X^TY = y_{\cdots}\mathbb{1} + X_\alpha(X_\alpha^TX_\alpha)^{-1}X_\alpha^TY + X_\beta(X_\beta^TX_\beta)^{-1}X_\beta^TY + X_\omega(X_\omega^TX_\omega)^{-1}X_\omega^TY$

$\hat{y} = P_\Omega(y)$; $P_{W_\alpha}(y)$, $P_{W_\beta}(y)$, $P_{W_\omega}(y)$

- Estimation :

$$
\begin{array}{ccccccccccc}
y_{ijl} & = & \hat{\eta} & + & \hat{\alpha}_i & + & \hat{\beta}_j & + & \hat{\omega}_{ij} & + & r_{ijl} \\
\downarrow y & = & \bar{y}_{\cdots} & + & (\bar{y}_{i\cdot\cdot} - \bar{y}_{\cdots}) & + & (\bar{y}_{\cdot j\cdot} - \bar{y}_{\cdots}) & + & (\bar{y}_{ij\cdot} - \bar{y}_{i\cdot\cdot} - \bar{y}_{\cdot j\cdot} + \bar{y}_{\cdots}) & &
\end{array}
$$

$P_{W_0}(y)$; $P_{W_\alpha}(y) \equiv a$; $P_{W_\beta}(y) \equiv b$; $P_{W_\omega}(y) \equiv c$

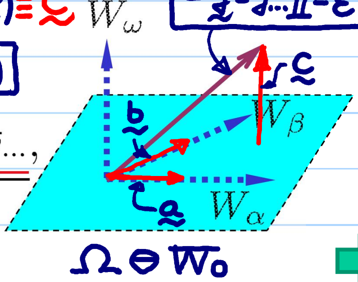
$+ (y_{ijl} - \bar{y}_{ij\cdot}),$ $P_{\Omega^\perp}(y) \equiv \hat{\varepsilon}$

$\boxed{\Omega \equiv (W_0 \oplus W_\alpha \oplus W_\beta \oplus W_\omega)}$

$\boxed{X\hat{\gamma} - y_{\cdots}\mathbb{1} = y - y_{\cdots}\mathbb{1} - \hat{\varepsilon}}$

$\boxed{\text{check LNp.3-4~5}} \rightarrow$
$\boxed{\begin{array}{l} P_{W_0\oplus W_\alpha}(y) = \bar{y}_{i\cdot\cdot} \\ P_{W_0\oplus W_\beta}(y) = \bar{y}_{\cdot j\cdot} \\ P_{W_0\oplus W_\alpha\oplus W_\beta\oplus W_\omega}(y) = \bar{y}_{ij\cdot} \end{array}}$

where $\hat{\eta} = \bar{y}_{\cdots}$, $\hat{\alpha}_i = \bar{y}_{i\cdot\cdot} - \bar{y}_{\cdots}$, $\hat{\beta}_j = \bar{y}_{\cdot j\cdot} - \bar{y}_{\cdots}$,
$\hat{\omega}_{ij} = \bar{y}_{ij\cdot} - \bar{y}_{i\cdot\cdot} - \bar{y}_{\cdot j\cdot} + \bar{y}_{\cdots}$,
$r_{ijl} = y_{ijl} - \bar{y}_{ij\cdot}$.



ANOVA → "whether there exist difference" problem

(exercise) check (P4) in LNp.2-36 & LNp.3-6

$$\|\underset{\sim}{y}\|^2 = \underbrace{\|P_{W_0}(\underset{\sim}{y})\|^2}_{\propto \bar{y}_{\cdot\cdot\cdot}^2} + \|P_{W_\alpha}(\underset{\sim}{y})\|^2 + \|P_{W_\beta}(\underset{\sim}{y})\|^2 + \|P_{W_\omega}(\underset{\sim}{y})\|^2 + \|P_{\Omega^\perp}(\underset{\sim}{y})\|^2$$

Table 9: ANOVA Table for Two-Way Layout

Source	Degrees of Freedom	Sum of Squares	MS = ?	E(MS) = ?
A	$I-1$	$nJ\sum_{i=1}^{I}(\bar{y}_{i\cdot\cdot}-\bar{y}_{\cdot\cdot\cdot})^2 = \|\underset{\sim}{a}\|^2$		
B	$J-1$	$nI\sum_{j=1}^{J}(\bar{y}_{\cdot j\cdot}-\bar{y}_{\cdot\cdot\cdot})^2 = \|\underset{\sim}{b}\|^2$		
$A\times B$	$(I-1)(J-1)$	$n\sum_{i=1}^{I}\sum_{j=1}^{J}(\bar{y}_{ij\cdot}-\bar{y}_{i\cdot\cdot}-\bar{y}_{\cdot j\cdot}+\bar{y}_{\cdot\cdot\cdot})^2 = \|\underset{\sim}{c}\|^2$		
residual	$IJ(n-1)$	$\sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{l=1}^{n}(y_{ijl}-\bar{y}_{ij\cdot})^2 = \|\hat{\underset{\sim}{\varepsilon}}\|^2$		
total	$IJn-\underbrace{1}_{B_0}$	$\sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{l=1}^{n}(y_{ijl}-\bar{y}_{\cdot\cdot\cdot})^2$		

規律$_A$
規律$_B$
規律$_{A\times B}$
隨機

$E(SS_{W_i}) = \|P_{W_i}X\underset{\sim}{\gamma}\|^2 + \sigma^2 \times \dim(W_i)$

What information in these quantities?

$\because W_0 \perp W_\alpha \perp W_\beta \perp W_\omega \perp \Omega^\perp$

anova(y~B_0+A+B+A×B) $\xleftrightarrow{\text{cf.}}$ anova(y~B_0+B+A+A×B)

identical results when orthogonality holds

→ also identical to drop-one ANOVA