Model

$y \sim B_0 + \underbrace{\text{media}}_{①\, I-1} + \underbrace{\text{plating}}_{②\, J-1} + \underbrace{\text{media}\times\text{plating}}_{②\, (I-1)(J-1)} + \varepsilon$

- Model : μ_{ij}

Functional form → $y_{ijl} = \eta + \alpha_i + \beta_j + \omega_{ij} + \varepsilon_{ijl}$, ← over-parameterized → Ok for the tests of ANOVA (5)

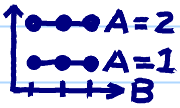
① ②I ③J ⑥I×J

$i = 1, \ldots, I$; $j = 1, \ldots, J$; $l = 1, \ldots, n$, where $E(y_{ijl}) = \mu_{ij}$ LM, LNp.8-17~18

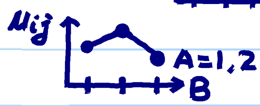
y_{ijl}	=	observation for the lth replicate of the ith level of factor A and the jth level of factor B,
α_i	=	ith main effect for A,
β_j	=	jth main effect for B,
ω_{ij}	=	(i,j)th interaction effect between A and B, and
ε_{ijl}	=	errors, independent $N(0, \sigma^2)$.

Q: What do α_i's & β_j's mean if model contain ω_{ij}'s?

(exercise)
$\bar{\mu}_{i\cdot} = 1\eta + X_\alpha \alpha$
$\bar{\mu}_{\cdot j} = 1\eta + X_\beta \beta$
(check LNp.4-17)

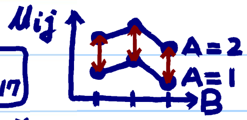


$y \sim B_0 + A + \varepsilon$ ①

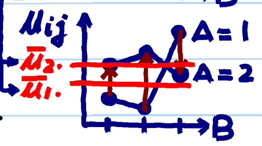


$y \sim B_0 + B + \varepsilon$ ②

check LNp.4-17



$y \sim B_0 + A + B + \varepsilon$ ① ②



$y \sim B_0 + A + B + A\times B + \varepsilon$ ②

Under sum codings,

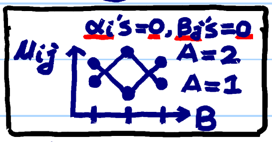
	$B(j)$				
$A(i)$	ω_{11}	ω_{12}	$\cdots$	ω_{1J}	$\Sigma=0$
	ω_{21}	ω_{22}	$\cdots$	ω_{2J}	$\Sigma=0$
	$\vdots$	$\vdots$	$\ddots$	$\vdots$	
	ω_{I1}	ω_{I2}	$\cdots$	ω_{IJ}	
	$\Sigma=0$				

$\alpha_I = -(\alpha_1 + \cdots + \alpha_{I-1}) \Rightarrow I \to I-1$

$\beta_J = -(\beta_1 + \cdots + \beta_{J-1}) \Rightarrow J \to J-1$

# of constraints $= I+J-1$ ←
$\omega_{iJ} = -(\omega_{i1} + \cdots + \omega_{iJ-1}),\ i = 1, \cdots, I$
$\omega_{Ij} = -(\omega_{1j} + \cdots + \omega_{I-1\,j}),\ j = 1, \cdots, J$
$\cancel{\sum_{i=1}^{I}\sum_{j=1}^{J}\omega_{ij} = 0}$

$\Rightarrow IJ \to IJ - (I+J-1) = (I-1)(J-1)$



Estimation

matrix form : $Y = X\gamma + \varepsilon$ sum codings

model matrix X

design matrix

$-\alpha_1$, $-\beta_1-\beta_2$, $-\omega_{11}-\omega_{12}$, $-\omega_{11}$, $-\omega_{12}$, $-\omega_{21}-\omega_{22}$ $=\omega_{11}+\omega_{12}$

$$\begin{array}{cc|ccc|ccccc|ccccccc|ccccc}
(i,j) & \mu_{ij} & \eta & \alpha_1 & & \alpha_2 & \beta_1 & & \beta_2 & & \beta_3 & \omega_{11} & & & \omega_{12} & & & \omega_{13} & \cancel{\omega_{21}} & & \cancel{\omega_{22}} & & \omega_{23} \\
1\ 1 & \to \mu_{11} & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1\ 2 & \to \mu_{12} & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1\ 3 & \to \mu_{13} & 1 & 1 & 1 & 0 & 0 & -1 & 0 & -1 & 1 & 0 & -1 & -1 & 0 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\
2\ 1 & \to \mu_{21} & 1 & 0 & -1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
2\ 2 & \to \mu_{22} & 1 & 0 & -1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 1 & 0 \\
2\ 3 & \to \mu_{23} & 1 & 0 & -1 & 1 & 0 & -1 & 0 & -1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & -1 & 1
\end{array}$$

span ↓ W_0 ↓span $W_\alpha(X_\alpha)$ ↓span $W_\beta(X_\beta)$ ↓span $W_\omega(X_\omega)$ ← $\{1\} \perp \{x_i\} \perp \{x_j\} \perp \{x_{ij}\}$ ⇐ ∵ balance

$x_{ij} = x_i \times x_j$

$x_i \perp x_j \perp x_{ij}$
x_i's : not orthogonal
x_j's : not orthogonal
x_{ij}'s : not orthogonal

$$\hat{\gamma} = (X^TX)^{-1}X^TY = \begin{bmatrix} \bar{y}_{\cdots} \\ (X_\alpha^TX_\alpha)^{-1}X_\alpha^TY \\ (X_\beta^TX_\beta)^{-1}X_\beta^TY \\ (X_\omega^TX_\omega)^{-1}X_\omega^TY \end{bmatrix} \begin{array}{l} \to \hat{\eta} \\ \to \hat{\alpha}_i\text{'s} \\ \to \hat{\beta}_j\text{'s} \\ \to \hat{\omega}_{ij}\text{'s} \end{array}$$ (exercise)

$X\hat{\gamma} = X(X^TX)^{-1}X^TY = \bar{y}_{\cdots}1 + \underbrace{X_\alpha(X_\alpha^TX_\alpha)^{-1}X_\alpha^TY}_{P_{W_\alpha}(y)} + \underbrace{X_\beta(X_\beta^TX_\beta)^{-1}X_\beta^TY}_{P_{W_\beta}(y)} + \underbrace{X_\omega(X_\omega^TX_\omega)^{-1}X_\omega^TY}_{P_{W_\omega}(y)}$

$\hat{y} = P_\Omega(y)$

- Estimation :

$$\begin{aligned}
y_{ijl} &= \hat{\eta} + \hat{\alpha}_i + \hat{\beta}_j + \hat{\omega}_{ij} + r_{ijl} \\
y &= \underbrace{\bar{y}_{\cdots}}_{P_{W_0}(y)} + \underbrace{(\bar{y}_{i\cdot\cdot} - \bar{y}_{\cdots})}_{P_{W_\alpha}(y) \equiv a} + \underbrace{(\bar{y}_{\cdot j\cdot} - \bar{y}_{\cdots})}_{P_{W_\beta}(y) \equiv b} + \underbrace{(\bar{y}_{ij\cdot} - \bar{y}_{i\cdot\cdot} - \bar{y}_{\cdot j\cdot} + \bar{y}_{\cdots})}_{P_{W_\omega}(y) \equiv c} \\
&\quad + \underbrace{(y_{ijl} - \bar{y}_{ij\cdot})}_{P_{\Omega^\perp}(y) \equiv \hat{\varepsilon}},
\end{aligned}$$

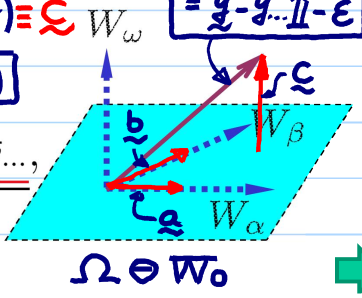
$\Omega \equiv (W_0 \oplus W_\alpha \oplus W_\beta \oplus W_\omega)$

where $\hat{\eta} = \bar{y}_{\cdots}$, $\hat{\alpha}_i = \bar{y}_{i\cdot\cdot} - \bar{y}_{\cdots}$, $\hat{\beta}_j = \bar{y}_{\cdot j\cdot} - \bar{y}_{\cdots}$,
$\hat{\omega}_{ij} = \bar{y}_{ij\cdot} - \bar{y}_{i\cdot\cdot} - \bar{y}_{\cdot j\cdot} + \bar{y}_{\cdots}$,
$r_{ijl} = y_{ijl} - \bar{y}_{ij\cdot}$.

check LNp.3-4~5

$P_{W_0 \oplus W_\alpha}(y) = \bar{y}_{i\cdot\cdot}$
$P_{W_0 \oplus W_\beta}(y) = \bar{y}_{\cdot j\cdot}$
$P_{W_0 \oplus W_\alpha \oplus W_\beta \oplus W_\omega}(y) = \bar{y}_{ij\cdot}$

$X\hat{\beta} - \bar{y}_{\cdots}1 = y - \bar{y}_{\cdots}1 - \hat{\varepsilon}$



ANOVA → "whether there exist difference" problem

(exercise) check (P4) in LNp.2-36 & LNp.3-6

$\|y\|^2 = \underbrace{\|P_{W_0}(y)\|^2}_{\propto \bar{y}_{\cdot\cdot\cdot}^2} + \|P_{W_\alpha}(y)\|^2 + \|P_{W_\beta}(y)\|^2 + \|P_{W_\omega}(y)\|^2 + \|P_{\Omega^\perp}(y)\|^2$

Table 9: ANOVA Table for Two-Way Layout

Source	Degrees of Freedom	Sum of Squares	MS = ?	E(MS) = ?
A	$I-1$	$nJ\sum_{i=1}^{I}(\bar{y}_{i\cdot\cdot}-\bar{y}_{\cdot\cdot\cdot})^2 = \|\hat{\alpha}\|^2$		
B	$J-1$	$nI\sum_{j=1}^{J}(\bar{y}_{\cdot j\cdot}-\bar{y}_{\cdot\cdot\cdot})^2 = \|\hat{\beta}\|^2$		
$A\times B$	$(I-1)(J-1)$	$n\sum_{i=1}^{I}\sum_{j=1}^{J}(\bar{y}_{ij\cdot}-\bar{y}_{i\cdot\cdot}-\bar{y}_{\cdot j\cdot}+\bar{y}_{\cdot\cdot\cdot})^2 = \|\hat{\gamma}\|^2$		
residual	$IJ(n-1)$	$\sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{l=1}^{n}(y_{ijl}-\bar{y}_{ij\cdot})^2 = \|\hat{\varepsilon}\|^2$		
total	$IJn-1$	$\sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{l=1}^{n}(y_{ijl}-\bar{y}_{\cdot\cdot\cdot})^2$		

規律$_A$
規律$_B$
規律$_{A\times B}$
隨機

B_0

$\because W_0 \perp W_\alpha \perp W_\beta \perp W_\omega \perp \Omega^\perp$

$E(SS_{W_i}) = \|P_{W_i} X\gamma\|^2 + \sigma^2 \times \dim(W_i)$

What information in these quantities?

anova(y ~ B₀ + A + B + A×B) $\xleftrightarrow{\text{cf.}}$ anova(y ~ B₀ + B + A + A×B)

identical results when orthogonality holds

→ also identical to drop-one ANOVA