Paired Comparison Design (Sewage Experiment)

- **Objective** : To compare two methods MSI and SIB for determining chlorine content in sewage effluents; y = residual chlorine reading.

★ response : reading
★ treatment factor : method (qualitative) 2 levels – MSI, SIB
★ block factor sample (qualitative) 8 levels – 1, 2, ⋯, 8 block size = 2
★ Exp'tal unit = ? 16 EUs
★ Design matrix

Table 1: Residual Chlorine Readings, Sewage Experiment

$\tau_2-\tau_1 = (\alpha_i+\tau_2)-(\alpha_i+\tau_1)$

$y_{SIB,i}-y_{MSI,i}$ (within block comparison)

Sample	Method: MSI	Method: SIB	d_i
1	0.39	0.36	−0.03
2	0.84	1.35	0.51
3	1.76	2.56	0.80
4	3.35	3.92	0.57
5	4.69	5.35	0.66
6	7.70	8.33	0.63
7	10.52	10.70	0.18
8	10.92	10.91	−0.01

blocks

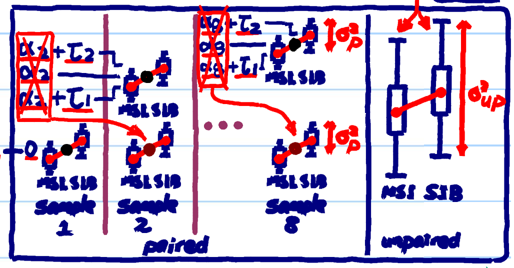
α_i's also removed in $\bar{y}_{SIB.}-\bar{y}_{MSI.}$ $(=\bar{d})$ but α_i's still influence $\hat{\sigma}^2$

Successful blocking
btwn-block variation : large
within-block variation : small

method	sample	y
MSI	1	0.39
SIB	1	0.36
MSI	2	0.84
SIB	2	1.35
⋮	⋮	⋮
MSI	8	10.92
SIB	8	10.91

Data matrix

α_i's still in them



- **Experimental Design** : Eight samples were collected at different doses and contact times. Two methods were applied to each of the eight samples. It is a *paired comparison* design because the pair of treatments are applied to the same samples (or units).

Paired Comparison Design vs. Unpaired Design

one-sample problem : $z_1, \cdots, z_n$ iid $\sim N(\mu, \sigma^2)$, $H_0: \mu = \mu_0$ ← a known constant

$\mu \overset{?}{=} \mu_0$

- *Paired Comparison Design* : Two treatments are randomly assigned to each *block* of *two* units. Can eliminate block-to-block variation and is effective if such variation is large. Examples: pairs of twins, eyes, kidneys, left and right feet. (Subject-to-subject variation much larger than within-subject variation). ← within-block variation

btwn block variation

For block effects, we are not interested in their estimation or testing. But, if we ignore block effects (either before or after exp't), the analysis of treatment effects suffer.

$Y = X_1B_1 + X_2B_2 + \varepsilon \quad \varepsilon^*$

$\to X_1B_1 + \cancel{H_1X_2B_2} + (I-H_1)X_2B_2 + \varepsilon$

- *Unpaired Design* : Each treatment is applied to a *separate* set of units, or called the *two-sample* problem. Useful if pairing is unnecessary; also it has more degrees of freedom for error estimation (see LNp. 4-4).

btwn block variation ≈ 0 ⇒ homogeneous EUs

★ Exp'tal units : water samples

unpaired ← one factor exp't

$\boxed{EU1}$ • • • $\boxed{EU16}$ ← 16 EUs : homogeneous

r MSI, SIB r ← completely randomized → $\frac{16!}{8!8!}$

paired ← 2 factor exp't

2^8 ← restrictedly randomized

sample 1: EU11, EU12 (MSI, SIB) • • • sample 8: EU81, EU82 (MSI, SIB)

16 EUs : hetrogeneous btwn samples, homogeneous within samples

2-sample problem : 1-way layout of a 2-level factor
y_{1j}'s iid $\sim N(\mu_1, \sigma^2)$
y_{2j}'s iid $\sim N(\mu_2, \sigma^2)$
$H_0: \mu_1 = \mu_2$

$\mu_1 \overset{?}{=} \mu_2$

Paired t tests

conceptual model: $y \sim B_0 + \text{sample} + \text{method} + \varepsilon$ ... (*)

7 parameter (sample), 1 parameter (method)

(main-effect model, no interaction btwn block & treatment factors, why?)

- **Paired t test :** Let y_{i1}, y_{i2} be the responses of treatments 1 and 2 for unit i, $i = 1, \ldots, N$. Let $d_i = y_{i2} - y_{i1}$, $\bar{d}$ and s_d^2 the sample mean and variance of d_i.

$\boxed{\bar{y}_{\cdot 2} - \bar{y}_{\cdot 1}}$

$$t_{paired} = \frac{\bar{d} - 0}{s_d/\sqrt{N}}$$

$\boxed{s_d^2 = \sum (d_i - \bar{d})^2 / N-1}$

one-sample problem
$d_1, \cdots, d_N$ iid $\sim N(\mu, \sigma^2)$
$H_0 : \mu = 0$ (effect of method)
0 $\quad \bar{d}$
$Var(\bar{d}) = \sigma^2 / N$

# of blocks

EUs → ☐ → y, (z)

Always put block factors before treatment factors in sequential ANOVA (why?)

F-test for "method"
Ω : (*)
ω : $B_0 + \text{sample} + \varepsilon$
both models contain "sample". The variation caused by "sample" is removed from ε.

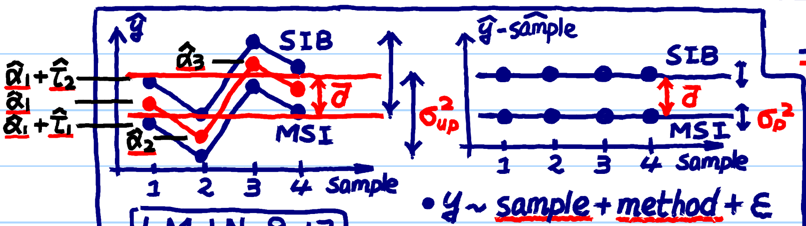
The two treatments are declared significantly different at level α if

$$|t_{paired}| > t_{N-1, \alpha/2}. \tag{1}$$

(exercise, use *) $(t_{paired})^2 \stackrel{?}{=} F = \dfrac{\|P_{\Omega \ominus \omega}(y)\|^2/1}{RSS_\Omega / df_\Omega}$, with $\|P_{\Omega \ominus \omega}(y)\|^2/1 = N\bar{d}^2/2$ and $RSS_\Omega/df_\Omega = s_d^2/2$

*
$y_{ij} = \hat{\eta} + \hat{\alpha}_i + \hat{\tau}_j + \hat{\varepsilon}_{ij}$
$\hat{\varepsilon}_{i1} = -\hat{\varepsilon}_{i2}$
$d_i = \hat{\tau}_2 - \hat{\tau}_1 + (\hat{\varepsilon}_{i2} - \hat{\varepsilon}_{i1})$
$\bar{d} = \hat{\tau}_2 - \hat{\tau}_1$
$d_i - \bar{d} = \hat{\varepsilon}_{i2} - \hat{\varepsilon}_{i1} = 2\hat{\varepsilon}_{i2}$

total df : 2N
df for intercept : 1
df for treatment factor : 1
df for block factor : N-1
df for residuals : 2N-1-1-(N-1) = N-1



LM, LNp. 8-17

- $y \sim \text{sample} + \text{method} + \varepsilon$
- $y \sim \text{method} + \varepsilon^*$

$\Omega \ominus \omega = \text{span}\{\underline{v}\}$

$\underline{v} = \begin{bmatrix} -1 \\ \vdots \\ -1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$ (y_{i1}'s, y_{i2}'s)

Unpaired t tests

(one-way layout, 2-sample problem)

conceptual model: $y \sim B_0 + \text{method} + \varepsilon'$ ... (Δ)
1 parameter

- **Unpaired t test :** The unpaired t test is appropriate if ___ we randomly choose N of the $2N$ units to receive one treatment and assign the remaining N units to the second treatment. Let $\bar{y}_i$ and s_i^2 be the sample mean and sample variance for the ith treatment, $i = 1$ and 2. Define

2-sample model in LNp. 2
$\mu_1 \quad \mu_2$
$\bar{y}_1 \quad \bar{y}_2$

F-test for method
Ω : (Δ)
ω : $y = B_0 + \varepsilon'$

$$t_{unpaired} = (\bar{y}_2 - \bar{y}_1) / \sqrt{(s_2^2/N) + (s_1^2/N)}.$$

$\bar{y}_2 - \bar{y}_1 = \bar{d}$; $(s_2^2/N) + (s_1^2/N) \equiv s^2/N$

$Var(\bar{y}_2 - \bar{y}_1) = \sigma_2^2/N + \sigma_1^2/N$

(exercise) $(t_{unpaired})^2 = F = \dfrac{\|P_{\Omega \ominus \omega}(y)\|^2/1}{RSS_\Omega/df_\Omega}$, $RSS_\Omega/df_\Omega = s^2/2$

$\Omega \ominus \omega = \text{span}\{\underline{v}\}$ (given in LNp. 4-3)

total df : 2N
df for intercept : 1
df for treatment factor : 1
df for residuals : 2N-1-1 = 2N-2

The two treatments are declared significantly different at level α if

$$|t_{unpaired}| > t_{2N-2, \alpha/2}. \tag{2}$$

- Note that the degrees of freedom in (1) and (2) are $N - 1$ and $2N - 2$ respectively. Unpaired t-test has more df's, but make sure that the unit-to-unit variation is under control (if this method is to be used).

$\boxed{t_{2N-2, \alpha/2} < t_{N-1, \alpha/2}}$

- If block effects significant (large btwn block variation) model (*) is better $\Rightarrow \hat{\sigma}^2_{(\Delta)} \gg \hat{\sigma}^2_{(*)}$
- If block effects insignificant (btwn block variation ≈ 0) model (Δ) is better ($\Rightarrow \hat{\sigma}^2_{(\Delta)} \approx \hat{\sigma}^2_{(*)}$) $\Rightarrow$ $t_{unpaired}$ has high power, and we can get better (accurate) estimate of σ^2 under (Δ)