Experimental Design and Analysis HW05 Solution

Problem 1

*Since each run takes about 15 minutes and there are 4 hours of daylight per day, we can conduct 16 runs per day and 64 runs over four consecutive days. The experiment has four two-level treatment factors (**shields**, **knob a**, **knob b**, **knob c**), giving $2^4 = 16$ level combinations. Because sunlight may vary across days, we treat **day** as a 4-level block factor and run all 16 combinations once per day (block size $= 16$).*

To reduce within-day time effects, we divide each day into 8 time slots and, for each knob setting, measure the two shields consecutively within the same slot. The order of Shield I and Shield II should be arranged using balanced randomization across the four days.

*Based on the discussion above, we adopt a **split-plot design** with*

- *3 whole-plot factors:* ***knob a**, **knob b**, **knob c*** (*whole-plot EU: 30 minutes*)
- *1 sub-plot factor:* ***shields*** (*sub-plot EU: 15 minutes*)
- *1 block factor:* ***day***

All knob setting combinations are as follows:

Level Combination	***Knob a***	***Knob b***	***Knob c***
1	+	+	+
2	+	+	−
3	+	−	+
4	+	−	−
5	−	+	+
6	−	+	−
7	−	−	+
8	−	−	−

The four-day experimental plan is as follows (the run order within each day should be randomized):

Run	***Day 1***	***Day 2***	***Day 3***	***Day 4***
1	$(1, I)$, $(1, II)$	$(1, I)$, $(1, II)$	$(1, II)$, $(1, I)$	$(1, II)$, $(1, I)$
2	$(2, II)$, $(2, I)$	$(2, I)$, $(2, II)$	$(2, II)$, $(2, I)$	$(2, I)$, $(2, II)$
3	$(3, I)$, $(3, II)$	$(3, II)$, $(3, I)$	$(3, I)$, $(3, II)$	$(3, II)$, $(3, I)$
4	$(4, II)$, $(4, I)$	$(4, II)$, $(4, I)$	$(4, I)$, $(4, II)$	$(4, I)$, $(4, II)$
5	$(5, I)$, $(5, II)$	$(5, II)$, $(5, I)$	$(5, II)$, $(5, I)$	$(5, I)$, $(5, II)$
6	$(6, II)$, $(6, I)$	$(6, I)$, $(6, II)$	$(6, I)$, $(6, II)$	$(6, II)$, $(6, I)$
7	$(7, I)$, $(7, II)$	$(7, I)$, $(7, II)$	$(7, II)$, $(7, I)$	$(7, II)$, $(7, I)$
8	$(8, II)$, $(8, I)$	$(8, II)$, $(8, I)$	$(8, I)$, $(8, II)$	$(8, I)$, $(8, II)$

The corresponding statistical model is:

$$y = \eta + \tau + \alpha + \beta + \alpha\beta + \varepsilon^{\mathcal{W}} + \varepsilon^{\mathcal{S}}$$

$$, \textit{ where } \begin{cases} \tau & \textit{is the block effect} \\ \alpha & \textit{are all the factorial effects of the whole-plot factors} \\ \beta & \textit{is the main effect of the sub-plot factor} \\ \alpha\beta & \textit{are the interactions between the whole-plot factors and the sub-plot factors} \\ \varepsilon^{\mathcal{W}} & \textit{are the whole-plot errors} \\ \varepsilon^{\mathcal{S}} & \textit{is the sub-plot error} \end{cases}$$

Problem 2

The table below reproduces Table 4.9 from page 178 of the textbook, which shows that the largest $|t_{PSE}|$, equal to 1.87, is due to the main effect of factor B.

Effect	$\mathbf{ln(s^2)}$
A	0.25
B	1.87
C	1.78
D	0.89
AB	0.71
AC	0.41
AD	0.46
BC	1.27
BD	0.16
CD	1.35
ABC	0.51
ABD	0.67
ACD	0.00
BCD	0.05
$ABCD$	1.63

With $I = 15$, we consult the table in Appendix H and apply linear interpolation:

$$\frac{(p\ value) - 0.07}{1.87 - 1.93} = \frac{0.08 - 0.07}{1.84 - 1.93}$$

The corresponding p-value is approximately 0.0767.

Therefore, at any significance level $\alpha \geq 0.0767$, the IER version of Lenth's method would identify at least one significant effect.

Problem 3

(a)

$$INT(B,C) = \bar{z}[(B,C) = (+,+);(-,-)] - \bar{z}[(B,C) = (+,-);(-,+)] = -23.5$$

The interaction effect estimate does not use the information in runs 9-11, because those runs are center points and therefore do not contribute to the ± 1 contrast.

(b)

Runs 9–11 can be used to estimate the underlying process variance σ^2 (i.e., the pure error variance), because they are replicated center-point runs carried out under the same factor settings. Thus, the variability among their responses reflects random experimental noise rather than factor effects. Using the three center-point responses 84, 87, and 81 with $\bar{y} = 84$, we estimate

$$s^2 = \frac{(84-84)^2 + (87-84)^2 + (81-84)^2}{3-1} = 9$$

(c)

The estimate of the standard error of the interaction effect $\hat{\theta}_{BC}$ is:

$$se(\hat{\theta}_{BC}) = \sqrt{\widehat{Var}(\hat{\theta}_{BC})} = \sqrt{\frac{4}{N}s^2} = \sqrt{4.5}$$

(d) (Supplementary)

The studentized maximum modulus test declares any $\hat{\theta}_i$ significant if $\dfrac{|\hat{\theta}_i|}{se(\hat{\theta}_i)} > |M|_{(I,\nu,\alpha)}$, where ν is the degrees of freedom associated with s^2.

The results are shown below:

```
y  <- data[1:8, 4]

A  <- rep(c(-1, 1), each = 1, times = 4)
B  <- rep(c(-1, 1), each = 2, times = 2)
C  <- rep(c(-1, 1), each = 4)

X   <- model.matrix(y ~ A * B * C)[, -1]
eff <- as.numeric(t(X) %*% y / 4)
names(eff) <- colnames(X)

names(eff) <- gsub(":", "", names(eff))
names(eff) <- trimws(names(eff))
```

```
yc <- data[9:11, 4]
s2 <- var(yc)
s  <- sqrt(s2)

N <- 8
se_eff <- 2 * s / sqrt(N)

tSMM <- round(abs(eff) / se_eff, 3)

Mcrit_001 <- 28.2
Mcrit_005 <- 12.44

color_cell <- function(val, thr){
  txt <- sprintf("%.3f", val)
  if (val >= thr) paste0("$\\textcolor{red}{", txt, "}$")
  else            paste0("$\\textcolor{black}{", txt, "}$")
}

make_row_color <- function(thr){
  sapply(tSMM, color_cell, thr = thr)
}

df <- data.frame(
  `$\\alpha$` = c("$0.01$", "$0.05$"),
  `$\\mathrm{Critical\\ value}\\quad\\quad$` =
    c(sprintf("%.2f", Mcrit_001), sprintf("%.2f", Mcrit_005)),
  rbind(make_row_color(11), make_row_color(11)),
  check.names = FALSE
)

df <- df[, c("$\\alpha$", "$\\mathrm{Critical\\ value}\\quad\\quad$",
             "A", "B", "C", "AB", "AC", "BC", "ABC")]

knitr::kable(df, row.names = FALSE, escape = FALSE)
```

α	Critical value	A	B	C	AB	AC	BC	ABC
0.01	28.20	0.707	0.236	0.236	0.236	0.707	11.078	0.236
0.05	12.44	0.707	0.236	0.236	0.236	0.707	11.078	0.236

From the SMM test, none of the seven factorial effects is significant at $\alpha = 0.05$ (and hence not at $\alpha = 0.01$).

To get the p-value of the largest effect, $\hat{\theta}_{BC}$, we interpolate between $|M|_{(7,2,0.1)} = 8.63$ and $|M|_{(7,2,0.05)} = 12.44$:

$$0.1 + (0.05 - 0.1) \times \frac{11.078 - 8.63}{12.44 - 8.63} \approx 0.06787402$$

(e)

The IER version of Lenth's method declares an effect $\hat{\theta}_i$ significant if $\left|t_{PSE,i}\right| = \left|\frac{\hat{\theta}_i}{PSE}\right|$ exceeds the critical value, where $PSE = 1.5 \times median_{\{|\hat{\theta}_i|<2.5\,s_0\}}|\hat{\theta}_i|$ and $s_0 = 1.5 \times median\,|\hat{\theta}_i|$.

The results are shown below:

```
s0   <- 1.5 * median(abs(eff))
trim <- 2.5 * s0
PSE  <- 1.5 * median(abs(eff)[abs(eff) < trim])

tPSE <- round(abs(eff / PSE), 3)

IER_001 <- 5.07
IER_005 <- 2.3

paint <- function(val, crit){
  txt <- sprintf("%.3f", val)
  if (val >= crit) paste0("$\\textcolor{red}{", txt, "}$")
  else             paste0("$\\textcolor{black}{", txt, "}$")
}
row_paint <- function(crit) sapply(tPSE, paint, crit = crit)

df <- data.frame(
  `$\\alpha$` = c("$0.01$", "$0.05$"),
  `$\\mathrm{Critical\\ value}\\quad\\quad$` = c(sprintf("%.2f", IER_001),
  sprintf("%.2f", IER_005)),rbind(row_paint(IER_001), row_paint(IER_005)),
  check.names = FALSE
)

df <- df[, c("$\\alpha$", "$\\mathrm{Critical\\ value}\\quad\\quad$",
             "A","B","C","AB","AC","BC","ABC")]

kable(df, row.names = FALSE, escape = FALSE)
```

α	Critical value	A	B	C	AB	AC	BC	ABC
0.01	5.07	2.000	0.667	0.667	0.667	2.000	31.333	0.667
0.05	2.30	2.000	0.667	0.667	0.667	2.000	31.333	0.667

When $I = 7$, the IER critical values are $IER_{0.01} = 5.07$ and $IER_{0.05} = 2.3$. Therefore, the BC effect is significant at $\alpha = 0.01$ (and hence at $\alpha = 0.05$).

This conclusion is completely different from that in (d).

Problem 4

(a)

They cannot distinguish between the effects of width and filler content because, with only the first two runs, the total degrees of freedom is 1, which is insufficient to separately estimate the main effects of width and filler content. Therefore, the two main effects are completely aliased based on the first two runs.

(b)

Yes, they can still use a 2^2 factorial design, as long as they choose the remaining two runs so that all four combinations of the two factors are conducted exactly once, which preserves balance and orthogonality for estimating the two main effects and the interaction.

The factor levels for the last two runs can be filled in as follows (assuming runs 1 and 2 are set as below):

Run	***Part Width***	***Filler Content***
1	+	+
2	+	−
3	−	+
4	−	−

(c)

No. They do not need to perform more than two additional runs to estimate all main effects provided they are willing to assume that all the (2-factor and 3-factor) interaction effects are negligible.

The experimental plan can be as follows, using a 2^{3-1} fractional factorial design generated by $A = BC$:

Run	***Part Width***	***Filler Content***	***Temperature***
1	+	+	+
2	−	+	−
3	+	−	−
4	−	−	+

However, if they are not willing to assume that all the interaction effects are negligible, then they need to use the following 2^3 full factorial design:

Run	***Part Width***	***Filler Content***	***Temperature***
1	+	+	+
2	+	+	−
3	+	−	+
4	+	−	−
5	−	+	+
6	−	+	−
7	−	−	+
8	−	−	−

Problem 5

(a)

We use a 2^3 full factorial design to construct the following conceptual model:

$$\mathbf{Z} = \mathbf{X}\beta + \mathcal{E},$$

We build a location model by setting $\mathbf{Z}_x = \bar{y}_x$, and a dispersion model by setting $\mathbf{Z}_x = ln(s_x^2)$, where x indexes the 8 treatment combinations.

The model matrix $\mathbf{X}$ is generated from the 2^3 full factorial design using the sum coding for the 3 factors A, B, C:

$$\mathbf{X} = \begin{array}{c} \begin{array}{cccccccc} \text{(Intercept)} & A & B & C & AB & AC & BC & ABC \end{array} \\ \begin{pmatrix} 1 & -1 & -1 & -1 & +1 & +1 & +1 & -1 \\ 1 & -1 & -1 & +1 & +1 & -1 & -1 & +1 \\ 1 & -1 & +1 & -1 & -1 & +1 & -1 & +1 \\ 1 & -1 & +1 & +1 & -1 & -1 & +1 & -1 \\ 1 & +1 & -1 & -1 & -1 & -1 & +1 & +1 \\ 1 & +1 & -1 & +1 & -1 & +1 & -1 & -1 \\ 1 & +1 & +1 & -1 & +1 & -1 & -1 & -1 \\ 1 & +1 & +1 & +1 & +1 & +1 & +1 & +1 \end{pmatrix} \end{array}$$

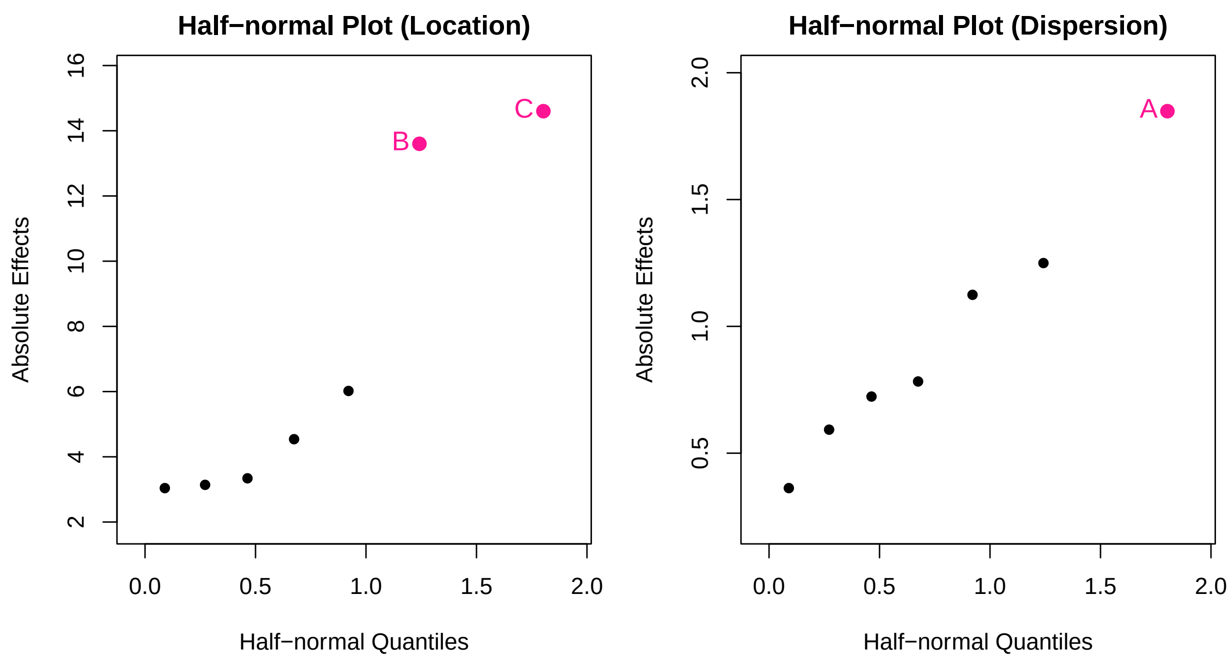
The factorial effects are estimated using regression analysis as follows:

```r
y_bar <- apply(data[,5:9], 1, mean)
ln_s_square <- apply(data[,5:9], 1, function(x){log(var(x))})
A <- rep(c(-1, 1), each = 4)
B <- rep(c(-1, 1), each = 2, 2)
C <- rep(c(-1,1), 4)

location_model <- lm(y_bar ~ A * B * C)
dispersion_model <- lm(ln_s_square ~ A * B * C)

name = c("A","B","C","AB","AC","BC","ABC")
location_effect = 2*coef(location_model)[-1]
dispersion_effect = 2*coef(dispersion_model)[-1]
table = cbind(name, round(location_effect, 4),round(dispersion_effect, 5))
colnames(table) = c("Effect", "$\\bar y$", "$ln(s^2)$")
kable(table, row.names = F)
```

Effect	$\bar{y}$	$ln(s^2)$
A	-6.02	-1.84838
B	-13.6	-0.59271
C	14.6	-0.72322
AB	4.54	1.24956
AC	3.14	1.12432
BC	3.04	0.78264
ABC	-3.34	-0.3623

Using half-normal plots to assess the significance of the location and dispersion effects:

```
halfnorm_pink <- function(x, outlier = 1, main = "Half-normal Plot",
                          hi_col = "deeppink",
                          xlab = "Half-normal Quantiles",
                          ylab = "Absolute Effects") {

  n <- length(x)

  # same logic as your code: sort absolute effects
  ord <- order(abs(x))
  y <- abs(x)[ord]
  lab <- names(x)[ord]

  # same quantile formula
  quan <- qnorm(0.5 + 0.5 * ((1:n) - 0.5) / n)

  # slightly expand range so labels stay inside the box
  xpad <- 0.08 * diff(range(quan))
  ypad <- 0.10 * diff(range(y))
  xlim <- c(min(quan) - xpad, max(quan) + xpad)
  ylim <- c(min(y) - ypad, max(y) + ypad)

  plot(quan, y, pch = 16, col = "black",
       main = main, xlab = xlab, ylab = ylab,
       xlim = xlim, ylim = ylim)

  # indices of points to label: EXACTLY the largest |effects|
  outlier <- min(outlier, n)
  idx <- (n - outlier + 1):n

  # highlight points
  points(quan[idx], y[idx], pch = 16, col = hi_col, cex = 1.4)

  # put labels inside the box (no bold)
  pos_vec <- ifelse(quan[idx] > mean(xlim), 2, 4)  # right-side points label to left
  text(quan[idx], y[idx], labels = lab[idx],
       col = hi_col, pos = pos_vec, offset = 0.35, cex = 1.2, font = 1)
}

par(mfrow = c(1,2), mar = c(4.2, 4.2, 2.2, 1.2))
halfnorm_pink(location_effect, outlier = 2, main = "Half-normal Plot (Location)")
halfnorm_pink(dispersion_effect,  outlier = 1, main = "Half-normal Plot (Dispersion)")
```



From the half-normal plots, the main effects of B and C appear significant in the location model. In contrast, none of the effects are clearly significant in the dispersion model, except that the main effect of A shows weak evidence of significance

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Using IER version of Lenth's method to assess the significance of the location and dispersion effects:

$$\textit{Declares an effect } \hat{\theta}_i \textit{ significant if } \left|t_{PSE,i}\right| = \left|\frac{\hat{\theta}_i}{PSE}\right| \textit{ exceeds the critical value,}$$

$$\textit{where } PSE = 1.5 \times median_{\{|\hat{\theta}_i|<2.5\,s_0\}} |\hat{\theta}_i| \textit{ and } s_0 = 1.5 \times median\, |\hat{\theta}_i|.$$

```
# --- IER version of Lenth's method (PSE) ---
pse_lenth <- function(eff){
  s0   <- 1.5 * median(abs(eff))
  trim <- 2.5 * s0
  1.5 * median(abs(eff)[abs(eff) < trim])
}

PSE_loc <- pse_lenth(location_effect)
PSE_dis <- pse_lenth(dispersion_effect)

table_abst <- cbind(
  name,
  round(abs(location_effect / PSE_loc), 2),
  round(abs(dispersion_effect / PSE_dis), 2)
)

colnames(table_abst) <- c("Effect", "$\\bar y$", "$\\ln(s^2)$")

knitr::kable(table_abst, row.names = FALSE, escape = FALSE)
```

Effect	$\bar{y}$	$\ln(s^2)$
A	0.88	1.57
B	2	0.5
C	2.14	0.62
AB	0.67	1.06
AC	0.46	0.96
BC	0.45	0.67
ABC	0.49	0.31

When $I = 7$, the IER critical value at $\alpha = 0.1$ is $IER_{0.1} = 1.71$ (see Appendix H of the textbook). Therefore, for the location model, the main effects of B and C are significant. For the dispersion model, none of the effects are significant, although the main effect of A is close to the IER critical value (i.e., it is borderline).

This conclusion is consistent with the half-normal plots: B and C stand out in the location plot, while no clear effects stand out in the dispersion plot except that A appears slightly more prominent than the others.

(b)

After the discussion of effect significance in part (a), we fit the following models:

$$\begin{cases} \text{Location model:} & \hat{\hat{y}} = 65.42 - 6.8B + 7.3C \\ \text{Dispersion model:} & \widehat{ln(s^2)} = 3.8714225 - 0.9241892A \end{cases}$$

Since B and C are adjustment factors, it is appropriate to use the two-step procedure.

- ***Step 1:*** *Set $A = +1$ (i.e., #5074) to minimize $\mathrm{Var}(z_x)$. The predicted variance is*

$$s^2 = \exp(3.8714225 - 0.9241892) = 19.05317.$$

- ***Step 2:*** *Choose B and C such that*

$$75 = 65.42 - 6.8B + 7.3C.$$

  For example, $(B, C) = (-1, 0.3808219)$, which corresponds to 800 rpm and 70.71233 mm/min.

Therefore, (#5074, 800 rpm, and 70.71233 mm/min) is a recommended set of optimal factor settings.

Problem 6

(a)

For the blocking scheme $\mathcal{B}_1^*: \{\mathbf{B_1 = 123},\ \mathbf{B_2 = 456},\ \mathbf{B_3 = 167}\}$, *the interactions confounded with the block effects are as follows:*

$$\begin{aligned}
&\mathbf{B_1 = 123}\\
&\mathbf{B_2 = 456}\\
&\mathbf{B_3 = 167}\\
&\mathbf{B_{12} = 123456}\\
&\mathbf{B_{13} = 2367}\\
&\mathbf{B_{23} = 1457}\\
&\mathbf{B_{123} = 23457}
\end{aligned}$$

We have $g(\mathcal{B}_1^*) = (0,0,3,2,1,1,0)$, *and therefore the order of estimability for* $\mathcal{B}_1^*$ *is 3.*

(b)

For the blocking scheme $\mathcal{B}_2^*: \{\mathbf{B_1 = 1234},\ \mathbf{B_2 = 1256},\ \mathbf{B_3 = 1357}\}$, *the interactions confounded with the block effects are as follows:*

$$\begin{aligned}
&\mathbf{B_1 = 1234}\\
&\mathbf{B_2 = 1256}\\
&\mathbf{B_3 = 1357}\\
&\mathbf{B_{12} = 3456}\\
&\mathbf{B_{13} = 2457}\\
&\mathbf{B_{23} = 2367}\\
&\mathbf{B_{123} = 1467}
\end{aligned}$$

We have $g(\mathcal{B}_2^*) = (0,0,0,7,0,0,0)$, *and therefore the order of estimability for* $\mathcal{B}_2^*$ *is 3.*

(c)

Comparing the blocking schemes $\mathcal{B}_1^*$ *and* $\mathcal{B}_2^*$, *we find that*

$$\min\left\{n \in \mathbb{N}^+ : g_n(\mathcal{B}_1^*) \neq g_n(\mathcal{B}_2^*)\right\} = 3$$

Since $g_3(\mathcal{B}_1^*) > g_3(\mathcal{B}_2^*)$, $\mathcal{B}_2^*$ *is superior to* $\mathcal{B}_1^*$ *under the minimum aberration criterion.*