# *Experimental Design and Analysis*
# *HW04 Solution*

## Problem 1

### (a)

*Since the data are collected according to a **RCBD** with 3 blocks of size 5(without replicates), the total degrees of freedom is 14=3 $\times$ 5 − 1 and the complete ANOVA table is shown below:*

| ***Source*** | ***Degrees of Freedom*** | ***Sum of Squares*** | ***Mean Squares*** |
|---|---|---|---|
| ***Block*** | 2 $(= 3 - 1)$ | 520 | 260 $(= 520/2)$ |
| ***Treatment*** | 4 $(= 5 - 1)$ | 498 | 124.5 $(= 498/4)$ |
| ***Residual*** | 8 $(= 14 - 2 - 4)$ | 40 | 5 $(= 40/8)$ |
| ***Total*** | 14 | 1058 $(= 520 + 498 + 40)$ | |

### (b)

*Do the multiple comparisons at level 0.01 with Tukey method. Testing*

$$\begin{cases} H_0^{(ij)} : \tau_i = \tau_j \\ H_1^{(ij)} : \tau_i \neq \tau_j \end{cases}$$

*with test statistic* $\mathscr{T}_{ij} = \frac{\overline{y}_{i\cdot} - \overline{y}_{j\cdot}}{\sqrt{5 \times \left(\frac{1}{3} + \frac{1}{3}\right)}}$ *and reject* $H_0^{(ij)}$ *if* $|\mathscr{T}_{ij}| > \frac{1}{\sqrt{2}} q_{(5,8,0.01)} \approx 4.68445034118$.

| $\lvert\mathscr{T}_{12}\rvert$ | $\lvert\mathscr{T}_{13}\rvert$ | $\lvert\mathscr{T}_{14}\rvert$ | $\lvert\mathscr{T}_{15}\rvert$ | $\lvert\mathscr{T}_{23}\rvert$ | $\lvert\mathscr{T}_{24}\rvert$ | $\lvert\mathscr{T}_{25}\rvert$ | $\lvert\mathscr{T}_{34}\rvert$ | $\lvert\mathscr{T}_{35}\rvert$ | $\lvert\mathscr{T}_{45}\rvert$ |
|---|---|---|---|---|---|---|---|---|---|
| 7.120393 | 0.547723 | 0.000000 | 6.024948 | 6.572671 | 7.120393 | 1.095445 | 0.547723 | 5.477226 | 6.024948 |

*From the table above, the following pairs of treatments are found to be significantly different:*

$$(1,2),\ (1,5),\ (2,3),\ (2,4),\ (3,5),\ (4,5)$$

made by 林宸緯、黃子維 助教

**(c)**

*In part (b), many pairs of treatments are found to be significantly different at level 0.01, which contradicts the null hypothesis of the F test. Hence, the F test tends to reject the null hypothesis at the same significance level.*

*Note that even at the same significance level, it is possible to find significant pairwise differences while the null hypothesis of F test is not rejected, since these two procedures are not theoretically equivalent.*

$\left(\textit{INTENTIONALLY LEFT BLANK}\right)$

# Problem 2

*Since* **power** *and* **ln(speed)** *are quantitative with three evenly spaced levels, we may fit a regression model:*

$$\mathbf{strength}_i = \eta + \alpha_\ell P_\ell(\mathbf{power}_i) + \beta_\ell P_\ell(\mathbf{ln(speed)}_i) + \alpha_q P_q(\mathbf{power}_i) + \beta_q P_q(\mathbf{ln(speed)}_i) + \gamma_{\ell\ell} P_\ell(\mathbf{power}_i) P_\ell(\mathbf{ln(speed)}_i) + \varepsilon_i$$

$$, \text{ where } \begin{cases} i \in \{1, \dots, 9\} & \\ \eta & \text{is the grand mean} \\ P_\ell \text{ aand } P_q & \text{are the normalized linear and quadratic contrasts, respectively} \\ \alpha_\ell \text{ and } \beta_\ell & \text{are the linear effect of } \mathbf{power} \text{ and } \mathbf{ln(speed)}, \text{ respectively} \\ \alpha_q \text{ and } \beta_q & \text{are the quadratic effect of } \mathbf{power} \text{ and } \mathbf{ln(speed)}, \text{ respectively} \\ \gamma_{\ell\ell} & \text{is the linear-by-linear interaction effect of } \mathbf{power} \text{ and } \mathbf{ln(speed)} \\ \varepsilon_i \overset{i.i.d.}{\sim} \mathcal{N}(0, \sigma^2) & \text{is the error term} \end{cases}$$

*Although we have enough degrees of freedom to include all of the interactions in our model, we only include $\gamma_{\ell\ell}$ here.*

*Thus, for the complete data*

$$\mathbf{strength} = \left(25.66, 28.00, 20.65, 29.15, 35.09, 29.79, 35.73, 39.56, 35.66\right)',$$

*the induced model matrix is*

$$\mathbf{X} = \begin{pmatrix} 1 & -1/\sqrt{2} & -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{6} & 1/2 \\ 1 & -1/\sqrt{2} & 0 & 1/\sqrt{6} & -2/\sqrt{6} & 0 \\ 1 & -1/\sqrt{2} & 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{6} & -1/2 \\ 1 & 0 & -1/\sqrt{2} & -2/\sqrt{6} & 1/\sqrt{6} & 0 \\ 1 & 0 & 0 & -2/\sqrt{6} & -2/\sqrt{6} & 0 \\ 1 & 0 & 1/\sqrt{2} & -2/\sqrt{6} & 1/\sqrt{6} & 0 \\ 1 & 1/\sqrt{2} & -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{6} & -1/2 \\ 1 & 1/\sqrt{2} & 0 & 1/\sqrt{6} & -2/\sqrt{6} & 0 \\ 1 & 1/\sqrt{2} & 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{6} & 1/2 \end{pmatrix}$$

```
##
## Call:
## lm(formula = strength ~ P_l.power + P_l.ln.speed + P_q.power +
##     P_q.ln.speed + P_l.power:P_l.ln.speed, data = data)
##
## Residuals:
##        1        2        3        4        5        6        7        8
##  0.50722  0.04556 -0.55278 -1.34111  0.56222  0.77889  0.83389 -0.60778
##        9
## -0.22611
##
## Coefficients:
##                         Estimate Std. Error t value Pr(>|t|)
## (Intercept)              31.0322     0.4038  76.853 4.86e-06 ***
## P_l.power                 8.6361     0.6994  12.348  0.00114 **
## P_l.ln.speed             -1.0465     0.6994  -1.496  0.23146
```

```
## P_q.power              -0.3810     0.6994  -0.545  0.62377
## P_q.ln.speed           -3.9001     0.6994  -5.577  0.01138 *
## P_l.power:P_l.ln.speed  2.4700     1.2114   2.039  0.13417
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 1.211 on 3 degrees of freedom
## Multiple R-squared:  0.9845, Adjusted R-squared:  0.9586
## F-statistic: 38.05 on 5 and 3 DF,  p-value: 0.006475
```

*We find that only the linear effect of* **power** *and quadratic effect of* **ln(speed)** *are significant.*
*Hence, we refit a model with the linear effect of* **power** *and both the linear and quadratic effects of* **ln(speed)** :

$$\mathbf{strength}_i = \eta + \alpha_\ell P_\ell(\mathbf{power}_i) + \beta_\ell P_\ell(\mathbf{ln(speed)}_i) + \beta_q P_q(\mathbf{ln(speed)}_i) + \varepsilon_i$$

```r
fit <- lm(strength ~ P_l.power + P_l.ln.speed + P_q.ln.speed, data)
summary(fit)
```

```
##
## Call:
## lm(formula = strength ~ P_l.power + P_l.ln.speed + P_q.ln.speed,
##     data = data)
##
## Residuals:
##       1       2       3       4       5       6       7       8       9
##  1.5867 -0.1100 -1.9433 -1.0300  0.8733  1.0900 -0.5567 -0.7633  0.8533
##
## Coefficients:
##              Estimate Std. Error t value Pr(>|t|)
## (Intercept)    31.032      0.493  62.942 1.92e-08 ***
## P_l.power       8.636      0.854  10.113 0.000162 ***
## P_l.ln.speed   -1.046      0.854  -1.225 0.274961
## P_q.ln.speed   -3.900      0.854  -4.567 0.006018 **
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 1.479 on 5 degrees of freedom
## Multiple R-squared:  0.9614, Adjusted R-squared:  0.9383
## F-statistic: 41.55 on 3 and 5 DF,  p-value: 0.0005869
```

*After doing parameter estimation, the fitted model is*

$$\widehat{\mathbf{strength}} = 31.0322 + 8.6361 P_\ell(\mathbf{power}) - 1.0465 P_\ell(\mathbf{ln(speed)}) - 3.9001 P_q(\mathbf{ln(speed)})$$

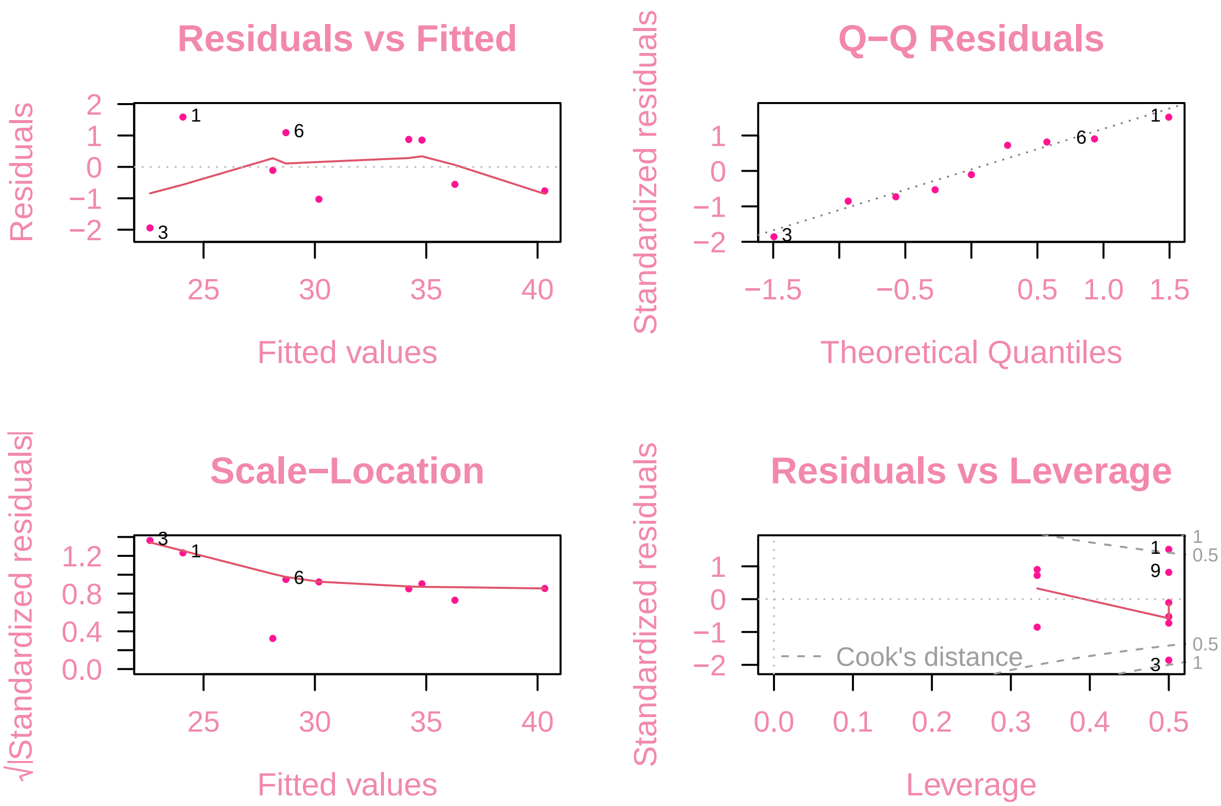
*The ANOVA table is:*

```
anova(fit)
```

```
## Analysis of Variance Table
##
## Response: strength
##               Df  Sum Sq Mean Sq  F value   Pr(>F)
## P_l.power      1 223.748 223.748 102.2746 0.000162 ***
## P_l.ln.speed  1   3.286   3.286   1.5018 0.274961
## P_q.ln.speed  1  45.633  45.633  20.8587 0.006018 **
## Residuals      5  10.939   2.188
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

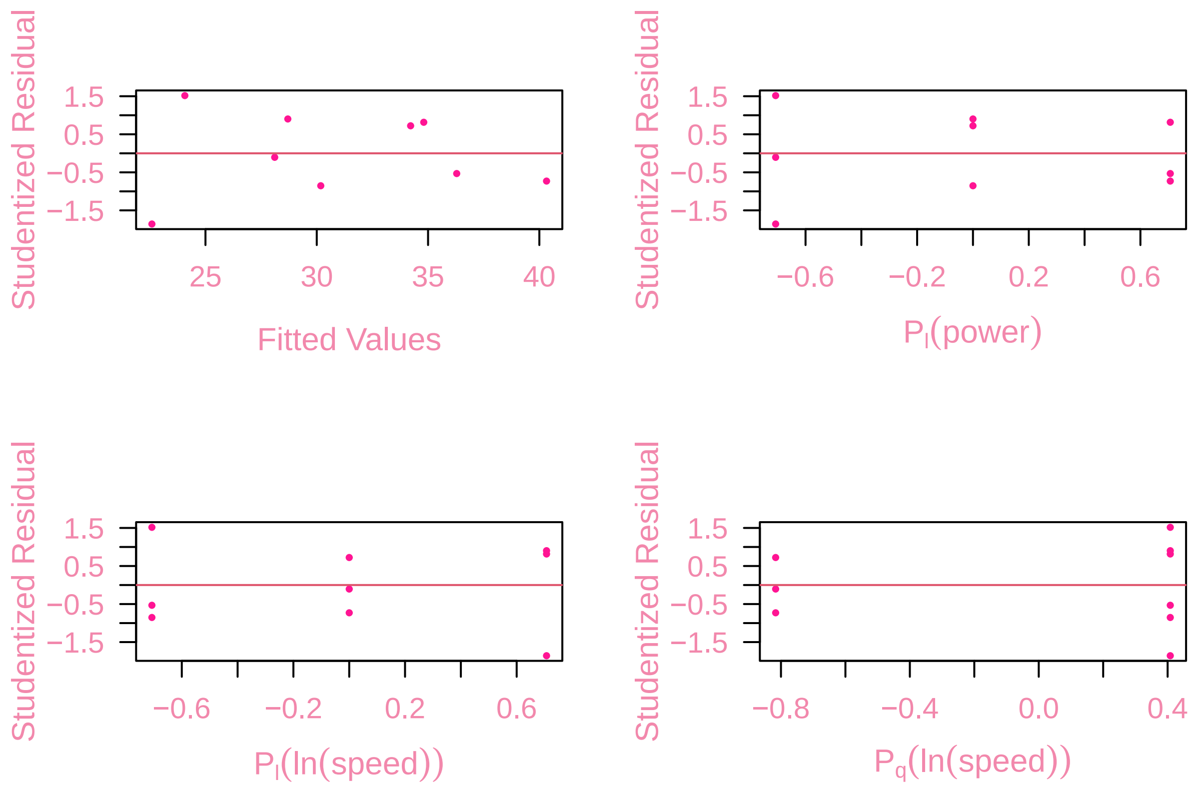
*The conclusions are the same as the t-tests above, since the F-tests and the t-tests are equivalent* ($F = t^2$ *in this case*).

*Do some diagnostics and analyze the residuals:*




*There are no obvious outliers and the QQ-plot shows that the residuals are approximately normally-distributed.*

*Then, plot the studentized residuals versus the fitted values,* $P_\ell(\textbf{power})$, $P_\ell(\textbf{ln(speed)})$ *and* $P_q(\textbf{ln(speed)})$ :




*Although some residual plots suggest non-constant variance and mean curvature, the model still fits the data well.*

$\left(\textit{INTENTIONALLY LEFT BLANK}\right)$

made by 林宸緯、黃子維 助教

## Problem 3

*It is a **4 × 4 Latin Square Design** with a treatment factor (**Gasoline Additive**) and two block factors (**Day, Time of Day**).*

---

***Pros:***

*(1) **Controls two nuisance sources simultaneously.** Because each additive appears once per day and once per time period, comparisons among additives are adjusted for both daily weather variation and time-of-day effects. This typically reduces unexplained variability and improves precision for estimating main effects of additives.*

*(2) **Orthogonality.** The design matrix of this Latin square design is of* $\mathbf{OA(16, 4^3, 2)}$*: treatment is orthogonal to each blocking factor. As a result, the estimated additive effects are not confounded with day-to-day or time-of-day variation, and the treatment main effects can be estimated independently of the block main effects. (Refer to page 319 of the textbook for details on orthogonal arrays.)*

*(3) **Small run size.** The Latin square achieves control of two four-level blocks with only* $4^2 = 16$ *runs, which is an efficient use of experimental effort. (If a **RCBD** was used,* $4^3 = 64$ *runs were needed.)*

---

***Cons:***

*(1) **Strong modeling assumption: no interactions with blocks.** The standard Latin square analysis assumes an additive model (Treatment + Day + Time) and cannot estimate interactions such as Treatment × Day or Treatment × Time. Thus, if an interaction effect does exist, then the parameter estimations will be biased.*

*(2) **No replication within a cell: limited error degrees of freedom.** Each Day× Time cell contains only one observation, so the residual degrees of freedom are small (here* $df_E = (4-1)(4-2) = 6$*). With small* $df_E$*, tests may have lower power and diagnostics are less stable.*

*(3) **Requires the same number of levels across factors.** A Latin square requires the same number of treatment levels and block levels, which reduces flexibility if the natural number of days or time points differs from the number of additives.*

---

*Supplementary material (link):*

A Library of Orthogonal Arrays (A crazy and interesting website—basically an encyclopedia for orthogonal arrays)

$\left( \textit{INTENTIONALLY LEFT BLANK} \right)$

# Problem 4

## (a)

$$\begin{cases} \textit{Treatment factor} & : \textbf{\textit{Shape}} \text{ (has 4 levels)} \\ \textit{Block factor} & : \textbf{\textit{Plate}} \text{ (has 4 levels, and the block size is 3)} \end{cases}$$

*Since every pair of distinct treatments occurs together in exactly two blocks* $(\lambda = 2)$, *this is a* ***BIBD***.

*Following the notation in the textbook, we have* $t = 4,\ b = 4,\ k = 3,\ r = 3$ *so that* $\lambda = \dfrac{r(k-1)}{t-1} = 2.$

## (b)

*Consider the statistical model* $(i \in \{A, B, C, D\}, j \in \{1, 2, 3, 4\})$ :

$$noise_{ij} = \eta + \alpha_i + \beta_j + \varepsilon_{ij}$$

$$\textit{where} \begin{cases} \eta & \textit{is the grand mean} \\ \alpha_i & \textit{is the treatment effect of } \textbf{\textit{shape}} \\ \beta_j & \textit{is the block effect of } \textbf{\textit{plate}} \\ \varepsilon_{ij} \overset{i.i.d.}{\sim} \mathcal{N}(0, \sigma^2) & \textit{is the error term} \end{cases}.$$

*Perform the ANOVA:*

```
##             Df Sum Sq Mean Sq F value Pr(>F)
## plate        3 0.3474 0.11579   8.455 0.0211 *
## shape        3 0.4651 0.15502  11.319 0.0115 *
## Residuals    5 0.0685 0.01369
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

*Since the p-value is quite small, shapes of the resistors do affect the current noise significantly.*

*As* $H_0$ *of the overall-F test is rejected, we do multiple comparisons by the Tukey method:*

```
##
##   Simultaneous Tests for General Linear Hypotheses
##
## Multiple Comparisons of Means: Tukey Contrasts
##
##
## Fit: aov(formula = noise ~ plate + shape, data = data)
##
## Linear Hypotheses:
##             Estimate Std. Error t value Pr(>|t|)
## B - A == 0 -0.45500    0.10135  -4.490   0.0236 *
```

```
## C - A == 0 -0.15625    0.10135  -1.542   0.4811
## D - A == 0 -0.50375    0.10135  -4.971   0.0157 *
## C - B == 0  0.29875    0.10135   2.948   0.1072
## D - B == 0 -0.04875    0.10135  -0.481   0.9601
## D - C == 0 -0.34750    0.10135  -3.429   0.0647 .
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
## (Adjusted p values reported -- single-step method)
```

*We conclude that the treatment pairs $(A, B)$ and $(A, D)$ are significantly different.*

$\left(\textit{INTENTIONALLY LEFT BLANK}\right)$

# Problem 5

*Consider the statistical model* ($i \in \{A, B\}, j \in \{1, \dots, 10\}$) :

$$y_{ij} = \eta + \alpha_i + \gamma x_{ij} + \varepsilon_{ij}$$

$$\text{where} \begin{cases} y_{ij} & \text{is the } \textbf{decrease in pressure} \text{ for the } j^{th} \text{ animal receiving the treatment } i \\ \eta & \text{is the grand mean} \\ \alpha_i & \text{is the treatment effect of } \textbf{substance} \\ \gamma & \text{is the regression coefficient for the covariate (} \textbf{initial pressure} \text{)} \\ x_{ij} & \text{is the } \textbf{initial pressure} \text{ for the } j^{th} \text{ animal receiving the treatment } i \\ \varepsilon_{ij} \overset{i.i.d.}{\sim} \mathcal{N}(0, \sigma^2) & \text{is the error term} \end{cases} .$$

```
##
## Call:
## lm(formula = decrease ~ initial + substance, data = data)
##
## Residuals:
##      Min       1Q   Median       3Q      Max
## -16.1497  -4.4518  -0.0662   4.7338  12.2199
##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept) -28.5373    12.6970  -2.248  0.03817 *
## initial       0.5175     0.1044   4.956  0.00012 ***
## substanceB    9.7427     3.5266   2.763  0.01331 *
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 7.788 on 17 degrees of freedom
## Multiple R-squared:  0.6272, Adjusted R-squared:  0.5834
## F-statistic:  14.3 on 2 and 17 DF,  p-value: 0.0002276
```

*The ANCOVA table is:*

```
## Analysis of Variance Table
##
## Response: decrease
##           Df  Sum Sq Mean Sq F value    Pr(>F)
## initial    1 1272.16 1272.16 20.9740 0.0002663 ***
## substance  1  462.92  462.92  7.6321 0.0133108 *
## Residuals 17 1031.12   60.65
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

*Through ANCOVA, we conclude that **substance A** and **substance B** are significantly different, as the p-value is small.*

*If we wrongly ignore the effect of the **initial pressure**, and use the following model to do ANOVA:*

$$y_{ij} = \eta + \alpha_i + \varepsilon_{ij}$$

```
##
## Call:
## lm(formula = decrease ~ substance, data = data)
##
## Residuals:
##    Min     1Q Median     3Q    Max
##  -18.2   -8.7    1.8   10.3   16.8
##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept)   33.200      3.743   8.871 5.46e-08 ***
## substanceB     7.000      5.293   1.323    0.203
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 11.83 on 18 degrees of freedom
## Multiple R-squared:  0.08857,    Adjusted R-squared:  0.03793
## F-statistic: 1.749 on 1 and 18 DF,  p-value: 0.2025
```

*The ANOVA table is:*

```
## Analysis of Variance Table
##
## Response: decrease
##           Df Sum Sq Mean Sq F value Pr(>F)
## substance  1  245.0  245.00  1.7492 0.2025
## Residuals 18 2521.2  140.07
```

*Then **substance** will no longer be identified as a significant factor.*

*Therefore, we must properly account for the effect of the covariate factor to avoid drawing misleading conclusions.*