I dummy variable, each for 1 diagonal component, e.g.,
- l_0: reference
- l_1, l_2, l_3: each coded as 1 in 1 dummy variable

- Generate a vector with I^2 components for an $(I+1)$-level nominal factor with the structure:

l_1	l_0	l_0
l_0	l_2	l_0
l_0	l_0	l_3

$$\text{QI-factor} = (l_1, l_0, l_0, l_0, l_2, l_0, l_0, l_0, l_3)^T$$

(not Y'; 1st, 2nd, 3rd)

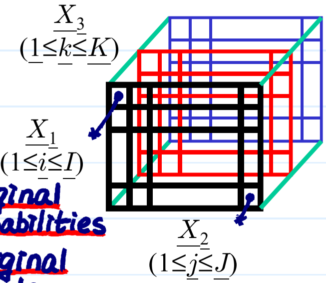
- $Y \sim X_1 + X_2 + \text{QI-factor} \equiv S_{\text{qindep2}}$
- Deviance-based/Pearson X^2 goodness-of-fit test for S_{qindep2}

Under S_{qindep2} model. ① $\hat{\pi}_{ii} = y_{ii}/y_{++} \Leftarrow \hat{\mu}_{ii} = y_{++}\hat{\pi}_{ii} = y_{ii}$ ⇒ diagonals y_{ii}'s have no contribution to deviance or X^2 ② add I more counts in Y, but also add I more parameters in $\eta = X\beta$ ⇒ cancelling out & same d.f.

❖ **Reading**: Faraway (2006, 1st ed.), 4.3

3 crossing covariates →

Three-Way Contingency Table



5/5

- The π's and Y's are defined in the same manner as in the 2-way table

π_{ijk} → marginal probabilities; Y_{ijk} → marginal totals (X_1, X_2, X_3)

- Poisson GLM approach to investigate how X_1, X_2, X_3 interact

(LNp.5-4) Recall Poisson GLM can be applied to SS1~4 but some information could be meaningless (LNp.5-13)

➢ Mutual independence (X_1, X_2, X_3 are independent)

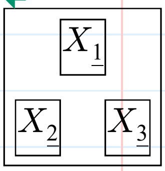
$Y_{ijk} \overset{\text{indep}}{\sim} P(\mu_{ijk})$, $\mu_{ijk} = t\,\pi_{ijk}$, log ↓ $X\beta = \eta_{ijk}$

joint = π marginals → ⦿ $\pi_{ijk} = \pi_{i++}\pi_{+j+}\pi_{++k} \Leftrightarrow P(X_1=i, X_2=j, X_3=k) = P(X_1=i)P(X_2=j)P(X_3=k)$

function of X_1 only

⦿ $\log(\pi_{ijk}) = \log(\pi_{i++}\pi_{+j+}\pi_{++k}) = \log(\pi_{i++}) + \log(\pi_{+j+}) + \log(\pi_{++k})$ (X_3, X_2)

$= \eta_{ijk} - \beta_0$



- $Y \sim X_1 + X_2 + X_3 \equiv S_1$

$X\hat{\beta} = \hat{\eta}$, log; $X^T\hat{\mu} \overset{\text{set}}{=} X^T Y$ (∵ canonical link); $\propto \hat{\pi}$

similar derivation as in LNp.5-6

⦿ The estimates of parameters in this model correspond only to the marginal totals y_{i++}, y_{+j+}, and y_{++k}

$= y_{+++}\hat{\pi}_{+j+}$; $= y_{+++} \times \hat{\pi}_{++k}$; $y_{+++}\hat{\pi}_{i++} =$; $\hat{\mu}_{ijk} = y_{+++}\hat{\pi}_{i++}\hat{\pi}_{+j+}\hat{\pi}_{++k}$

e.g., treatment, sum, ⋯, coding

- The coding we use will determine exactly how the parameters relate to the margin totals,

additive in η ↔ multiplicative in μ & π. If $\beta = \sum_i a_i \eta_i$ ($\beta = (X^TX)^{-1}X^T\eta$), then $e^\beta = \prod_i \mu_i^{a_i}$ ($\mu_i = e^{\eta_i}$) ✯

e.g., if $\beta = \eta_A - \frac{\eta_A + \eta_B + \eta_C}{3}$, $e^\beta = \frac{\mu_A}{\sqrt[3]{\mu_A\mu_B\mu_C}} = \frac{\pi_A}{\sqrt[3]{\pi_A\pi_B\pi_C}}$

if $\beta = \eta_B - \eta_A$, $e^\beta = \frac{\mu_B}{\mu_A} = \frac{\pi_B}{\pi_A}$

e.g., let β be an main effect of X_1 that codes i_1 and i_2 categories as **0** (reference) and 1

X_1	β_1	β_2	η	μ ($\propto \pi$)
A (reference)	0	0	$+0$	$\times e^0 = 1$
B	1	0	$+\beta_1$	$\times e^{\beta_1}$
C	0	1	$+\beta_2$	$\times e^{\beta_2}$

when X_2, X_3 are fixed

$$\Rightarrow e^{\hat{\beta}}/(1+e^{\hat{\beta}}) \overset{\checkmark}{=} \hat{\pi}_{i_2++}/(\hat{\pi}_{i_1++} + \hat{\pi}_{i_2++}) = \hat{p}_{i_1,i_2} \Rightarrow \text{logit}(\hat{p}_{i_1,i_2}) = \hat{\beta}$$

(cf. logistic)

$$= y_{i_2++}/(y_{i_1++} + y_{i_2++})$$

when all β's = 0, $e^0 : e^{\beta_1} : \cdots : e^{\beta_{I-1}} = 1 : 1 : \cdots : 1$

all its main effects are insignificant as in ANOVA

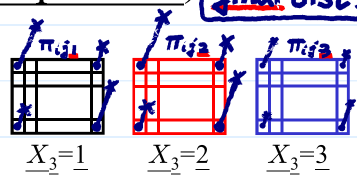
- Insignificant factor, say $X_1 \Rightarrow \pi_{1++} = \pi_{2++} = \ldots = \pi_{I++}$ (uniform marginal dist.)

➢ Joint independence ($\{X_1, X_2\}$ and X_3 are independent)

no need to be independent

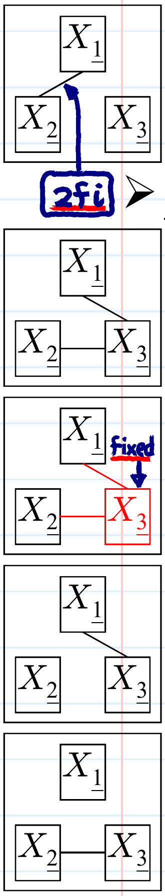
⦿ $\pi_{ijk} = \pi_{ij+} \times \pi_{++k} \Leftrightarrow \pi_{ij|k} = \pi_{ij+}$

$$\frac{\pi_{ijk}}{\pi_{++k}} = \frac{P(X_1=i, X_2=j, X_3=k)}{P(X_3=k)} = P(X_1=i, X_2=j \mid X_3=k) = \pi_{ij|k}$$



$\eta_{ijk} - \beta_0 =$ $\log(\pi_{ijk}) = \log(\pi_{ij+}\pi_{++k})$ — can be any function of X_1 & X_2

$= \log(\pi_{ij+}) + \log(\pi_{++k})$ — a 2-fi (2 factor interaction); function of X_3 only



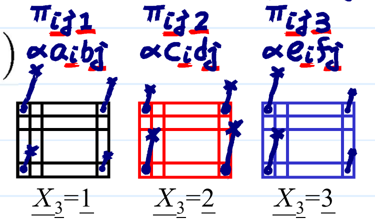
- $Y \sim X_1 + X_2 + X_1{:}X_2 + X_3 \equiv S_2\ (\supseteq S_1)$ — saturated model of X_1 & X_2; mutual independence ⇒ joint independence

➢ Conditional independence (X_1, X_2 are independent given X_3)

- $\pi_{ij|k}=\pi_{i+|k}\pi_{+j|k} \Leftrightarrow \pi_{ijk}=\pi_{i+k}\pi_{+jk}/\pi_{++k}$ — $\frac{\pi_{ijk}}{\pi_{++k}} = \frac{\pi_{i+k}}{\pi_{++k}} \times \frac{\pi_{+jk}}{\pi_{++k}}$; X_3 is fixed

$\log(\pi_{ijk}) = \log(\pi_{i+k}\pi_{+jk}/\pi_{++k})$

$\eta_{ijk} - \beta_0 = \log(\pi_{i+k}) + \log(\pi_{+jk}) - \log(\pi_{++k})$ — (X_1, X_3), (X_2, X_3), (X_3); common factor in 2fi's



- $Y \sim X_1 + X_1{:}X_3 + X_3 + X_2 + X_2{:}X_3 \equiv S_3$ — saturated model of X_1 & X_3; saturated model of X_2 & X_3

- Note that $S_3 \not\supseteq S_2$, but X_2 is jointly independent of $\{X_1, X_3\}$ implies that X_1, X_2 are independent given X_3 — joint indep. ⇒ conditional indep.

- **Q**: can this conditional independence imply independence between X_1 and X_2, i.e., $\pi_{ij+}=\pi_{i++}\pi_{+j+}$? (Ans: No. Check singular value decomposition in LNp.5-15)