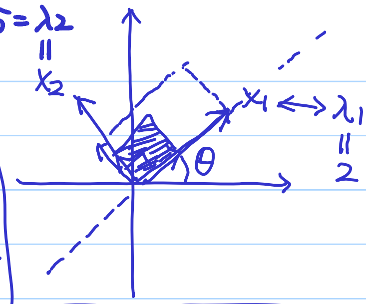
Definition 2A.31. Let $\mathbf{A}$ be a square matrix of dimension $k \times k$ and let λ be an eigenvalue of $\mathbf{A}$. If $\underset{(k\times 1)}{\mathbf{x}}$ is a *nonzero vector* ($\underset{(k\times 1)}{\mathbf{x}} \neq \underset{(k\times 1)}{\mathbf{0}}$) such that

$X=[x_1|\cdots|x_k]$

$AX=X\Lambda=[\lambda_1 x_1|\cdots|\lambda_k x_k]$

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$$

then $\mathbf{x}$ is said to be an *eigenvector (characteristic vector)* of the matrix $\mathbf{A}$ associated with the *eigenvalue* λ.



- uniqueness of eigenvectors $(A(cx)=cAx=c\lambda x=\lambda(cx))$
  - when all λ_i's are different.
  - some λ_i's are identical $\Rightarrow$ $\boxed{\text{if } \begin{matrix} Ax_1=\lambda_1 x_1 \\ Ax_2=\lambda_1 x_2 \end{matrix} \Rightarrow A(a_1x_1+a_2x_2) = a_1\lambda_1x_1+a_2\lambda_2x_2 = \lambda_1(a_1x_1+a_2x_2)}$
- eigenvectors corresponding to distinct eigenvalues are perpendicular (A: symmetric)
- determinant = product of eigenvalues
  - $|\mathbf{I}|=1$

$\boxed{cAx=c\cdot\lambda x=(c\lambda)x}$

$x_1^TAx_2 = x_1^T\lambda_2x_2 = \lambda_2(x_1^Tx_2)$
$(x_1^TAx_2)^T = x_2^TA^Tx_1 = x_2^TAx_1 = \lambda_1(x_2^Tx_1)$
$x_1^Tx_2=0$

- eigensystem

Let $\mathbf{A}$ be a $k \times k$ square $\boxed{\text{symmetric}}$ matrix. Then $\mathbf{A}$ has k pairs of eigenvalues and eigenvectors namely,

real

$$\lambda_1, \mathbf{e}_1 \quad \lambda_2, \mathbf{e}_2 \quad \cdots \quad \lambda_k, \mathbf{e}_k$$

The eigenvectors can be chosen to satisfy $1 = \mathbf{e}_1'\mathbf{e}_1 = \cdots = \mathbf{e}_k'\mathbf{e}_k$ and be mutually perpendicular. The eigenvectors are unique unless two or more eigenvalues are equal. $\{e_1,\cdots,e_k\}$ forms an orthogonal basis.

$P=[e_1|\cdots|e_k]$, $PP'=P'P=I$.

➢ trace

Definition 2A.28. Let $\mathbf{A} = \{a_{ij}\}$ be a $k \times k$ square matrix. The *trace* of the matrix $\mathbf{A}$, written $\mathrm{tr}(\mathbf{A})$, is the sum of the diagonal elements; that is, $\mathrm{tr}(\mathbf{A}) = \sum_{i=1}^{k} a_{ii}$.

Result 2A.12. Let $\mathbf{A}$ and $\mathbf{B}$ be $k \times k$ matrices and c be a scalar.

(a) $\mathrm{tr}(c\mathbf{A}) = c\,\mathrm{tr}(\mathbf{A})$

(b) $\mathrm{tr}(\mathbf{A} \pm \mathbf{B}) = \mathrm{tr}(\mathbf{A}) \pm \mathrm{tr}(\mathbf{B})$

(c) $\mathrm{tr}(\mathbf{AB}) = \mathrm{tr}(\mathbf{BA})$ $\Rightarrow \mathrm{tr}(A_1\cdots A_n)=\mathrm{tr}(A_2\cdots A_nA_1) = \mathrm{tr}(A_3\cdots A_nA_1A_2) = \cdots$

$\sum_{ij} a_{ij}b_{ji}$

(d) $\mathrm{tr}(\mathbf{B}^{-1}\mathbf{AB}) = \mathrm{tr}(\mathbf{A})$ $=\mathrm{tr}(ABB^{-1})$

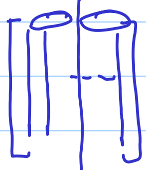
(e) $\mathrm{tr}(\mathbf{AA}') = \sum_{i=1}^{k}\sum_{j=1}^{k} a_{ij}^2$ $\Leftarrow$ generalization of length of a vector to matrix.

$=\mathrm{tr}(A'A)$

(f) $\mathrm{tr}(A)=\mathrm{tr}(P\Lambda P') = \mathrm{tr}(\Lambda P'P) = \mathrm{tr}(\Lambda)$ = sum of eigenvalues.

➢ partition of matrix
- 2×2 case

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

X (data set) $=$ 

$S_n = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}$. S_n, S_{11}, S_{22}: symmetric. S_{12}, S_{21}: not symmetric in general.

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} + \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} A_{11}+B_{11} & A_{12}+B_{12} \\ A_{21}+B_{21} & A_{22}+B_{22} \end{bmatrix}$$

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} A_{11}B_{11}+A_{12}B_{21} & A_{11}B_{12}+A_{12}B_{22} \\ A_{21}B_{11}+A_{22}B_{21} & A_{21}B_{12}+A_{22}B_{22} \end{bmatrix}$$

- block diagonal matrix: partition matrix **A** for which $\mathbf{A}_{ij}$=0 if $i \neq j$, e.g.,

$$A = \left[\begin{array}{c|c} A_{11} & 0 \\ \hline 0 & A_{22} \end{array}\right]$$

  - a block diagonal matrix is invertible iff each $\mathbf{A}_{ij}$=0 is invertible and

$$A^{-1} = \begin{bmatrix} A_{11}^{-1} & 0 \\ 0 & A_{22}^{-1} \end{bmatrix}$$

  - $|\mathbf{A}|=|\mathbf{A}_{11}|\times|\mathbf{A}_{22}|$
- these works with more than 2 by 2 partitioning

• positive definite matrix

➢ spectral decomposition and singular-value decomposition

Result 2A.14. *The Spectral Decomposition.* Let **A** be a $k \times k$ $\boxed{\text{symmetric}}$ matrix. Then **A** can be expressed in terms of its k eigenvalue–eigenvector pairs $(\lambda_i, \mathbf{e}_i)$ as

$P=[e_1,\cdots,e_k]$

$AP=P\Lambda$

$A=P\Lambda P^{-1}$

$=P\Lambda P' \rightarrow [\lambda_1 e_1 | \cdots | \lambda_k e_k]\begin{bmatrix} e_1' \\ \vdots \\ e_k' \end{bmatrix}$

$$\mathbf{A} = \sum_{i=1}^{k} \lambda_i \mathbf{e}_i \mathbf{e}_i' \rightarrow \text{real.}$$

$$\underset{(k\times k)}{\mathbf{A}} = \lambda_1 \underset{(k\times 1)(1\times k)}{\mathbf{e}_1 \mathbf{e}_1'} + \lambda_2 \underset{(k\times 1)(1\times k)}{\mathbf{e}_2 \mathbf{e}_2'} + \cdots + \lambda_k \underset{(k\times 1)(1\times k)}{\mathbf{e}_k \mathbf{e}_k'}$$

Result 2A.15. *Singular-Value Decomposition.* Let **A** be an $m \times k$ matrix of real numbers. Then there exist an $m \times m$ orthogonal matrix **U** and a $k \times k$ orthogonal matrix **V** such that

$$\mathbf{A} = \mathbf{U}\mathbf{\Lambda}\mathbf{V}'$$

where the $m \times k$ matrix $\mathbf{\Lambda}$ has (i,i) entry $\lambda_i \geq 0$ for $i = 1, 2, \ldots, \min(m,k)$ and the other entries are zero. The positive constants λ_i are called the *singular values* of **A**.

- $\mathbf{AV} = \mathbf{U\Lambda}$ $(\because A=U\Lambda V' \Rightarrow AV=U\Lambda V'V \Rightarrow AV=U\Lambda)$

  $\mathbf{A'U} = \mathbf{V\Lambda}$ $(\because A'=V\Lambda' U')$

  k-dim $\xrightarrow{A}$ m-dim

- $\underbrace{\mathbf{AA'}}_{m\times m} = \mathbf{U\Lambda^2U'}$ $(AA'=U\Lambda V'V\Lambda'U')$ symmetric

  $\Rightarrow$ squared singular values of **A** are eigenvalues of **AA′** and columns of **U** are eigenvectors of **AA′**

  $\underbrace{\mathbf{A'A}}_{k\times k} = \mathbf{V\Lambda^2V'}$

  $\Rightarrow$ squared singular values of **A** are eigenvalues of **A′A** and columns of **V** are eigenvectors of **A′A**

➢ quadratic form $\rightarrow \mathbb{R}^k \rightarrow \mathbb{R}^1$

Definition 2A.32. A *quadratic form* $Q(\mathbf{x})$ in the k variables $x_1, x_2, \ldots, x_k$ is $Q(\mathbf{x}) = \mathbf{x}'\mathbf{A}\mathbf{x}$, where $\mathbf{x}' = [x_1, x_2, \ldots, x_k]$ and **A** is a $k \times k$ symmetric matrix.

$\mathbf{x}'\mathbf{A}\mathbf{x}$ spectral decomposition

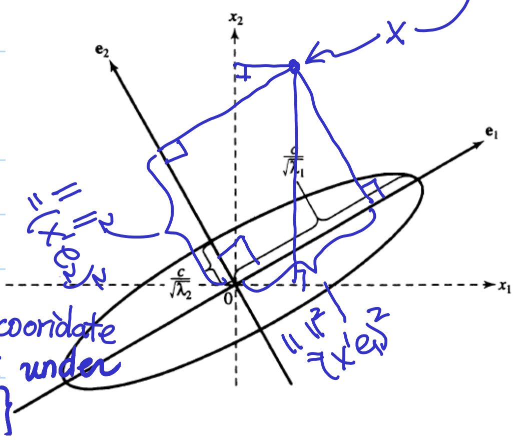
$= \mathbf{x}'(\lambda_1 \mathbf{e}_1 \mathbf{e}_1' + \lambda_2 \mathbf{e}_2 \mathbf{e}_2' + \cdots + \lambda_k \mathbf{e}_k \mathbf{e}_k')\mathbf{x}$

$= \lambda_1(\mathbf{x}'\mathbf{e}_1)(\mathbf{e}_1'\mathbf{x}) + \cdots + \lambda_k(\mathbf{x}'\mathbf{e}_k)(\mathbf{e}_k'\mathbf{x})$

$= \lambda_1(\underbrace{\mathbf{x}'\mathbf{e}_1}_{\text{inner product}})^2 + \cdots + \lambda_k(\mathbf{x}'\mathbf{e}_k)^2 = \lambda_1 y_1^2 + \cdots + \lambda_k y_k^2$, $y_i = x'e_i$

= squared length of the projection of x on e_1

$(y_1, \cdots, y_k)$: coordinate values of x under $\{e_1, \cdots, e_k\}$



➢ nonnegative definite and positive definite matrix

positive semi-definite

- When a $k \times k$ symmetric matrix $\mathbf{A}$ is such that

$$0 \le \mathbf{x}'\mathbf{A}\mathbf{x}$$

for all $\mathbf{x}' = [x_1, x_2, \ldots, x_k]$, both the matrix $\mathbf{A}$ and the quadratic form are said to be *nonnegative definite*.

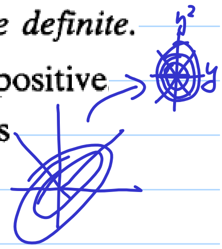
- When a $k \times k$ symmetric matrix $\mathbf{A}$ is such that

$$0 < \mathbf{x}'\mathbf{A}\mathbf{x} \quad (x=0 \Rightarrow x'Ax=0)$$

for all vectors $\mathbf{x} \neq \mathbf{0}$, $\mathbf{A}$ or the quadratic form is said to be *positive definite*.

- $\mathbf{A}$ is a positive definite matrix if and only if every eigenvalue of $\mathbf{A}$ is positive

  $\mathbf{A}$ is a nonnegative definite matrix if and only if all of its eigenvalues are greater than or equal to zero $(\sqrt{\lambda_1}y_1, \sqrt{\lambda_2}y_2, \cdots, \sqrt{\lambda_k}y_k)$



- For nonnegative definite or positive definite matrix

$$\mathbf{x}'\mathbf{A}\mathbf{x} = \sum \lambda_i y_i^2 = \sum (\sqrt{\lambda_i} y_i)^2 \qquad 1 = \lambda_1 y_1^2 = \lambda_2 y_2^2 = \cdots$$

- statistical distance and positive definite matrix $\Leftrightarrow \|y_1\| : \|y_2\| = \frac{1}{\sqrt{\lambda_1}} : \frac{1}{\sqrt{\lambda_2}}$.

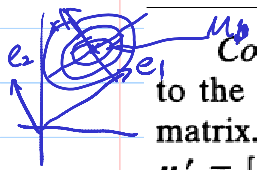
e.g. $\lambda_1 : \lambda_2 = 1:4$

$$(\text{distance})^2 = a_{11}x_1^2 + a_{22}x_2^2 + \cdots + a_{pp}x_p^2$$
$$+ 2(a_{12}x_1x_2 + a_{13}x_1x_3 + \cdots + a_{p-1,p}x_{p-1}x_p)$$

LNp.1-8

$$= [x_1, x_2, \ldots, x_p] \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ a_{21} & a_{22} & \cdots & a_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ a_{p1} & a_{p2} & \cdots & a_{pp} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix} = \mathbf{x}'\mathbf{A}\mathbf{x}$$

$\Rightarrow$ distance is determined from a positive definite quadratic form $\mathbf{x}'\mathbf{A}\mathbf{x}$.



Comment. Let the square of the distance from the point $\mathbf{x}' = [x_1, x_2, \ldots, x_p]$ to the origin be given by $\mathbf{x}'\mathbf{A}\mathbf{x}$, where $\mathbf{A}$ is a $p \times p$ symmetric positive definite matrix. Then the square of the distance from $\mathbf{x}$ to an arbitrary fixed point $\boldsymbol{\mu}' = [\mu_1, \mu_2, \ldots, \mu_p]$ is given by the general expression $(\mathbf{x} - \boldsymbol{\mu})'\mathbf{A}(\mathbf{x} - \boldsymbol{\mu})$.

If $p > 2$, the points $\mathbf{x}' = [x_1, x_2, \ldots, x_p]$ a constant distance $c = \sqrt{\mathbf{x}'\mathbf{A}\mathbf{x}}$ from the origin lie on hyperellipsoids $c^2 = \lambda_1(\mathbf{x}'\mathbf{e}_1)^2 + \cdots + \lambda_p(\mathbf{x}'\mathbf{e}_p)^2$, whose axes are given by the eigenvectors of $\mathbf{A}$. The half-length in the direction $\mathbf{e}_i$ is equal to $c/\sqrt{\lambda_i}$, $i = 1, 2, \ldots, p$, where $\lambda_1, \lambda_2, \ldots, \lambda_p$ are the eigenvalues of $\mathbf{A}$.

- square-root matrix

➢ Recall: matrix representation of spectral decomposition

Let $\mathbf{A}$ be a $k \times k$ positive definite matrix with the spectral decomposition $\mathbf{A} = \sum_{i=1}^{k} \lambda_i \mathbf{e}_i \mathbf{e}_i'$

Let the normalized eigenvectors be the columns of another matrix

$$\mathbf{P} = [\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_k]$$

Then

$$\underset{(k\times k)}{\mathbf{A}} = \sum_{i=1}^{k} \lambda_i \underset{(k\times 1)}{\mathbf{e}_i} \underset{(1\times k)}{\mathbf{e}_i'} = \underset{(k\times k)}{\mathbf{P}} \underset{(k\times k)}{\mathbf{\Lambda}} \underset{(k\times k)}{\mathbf{P}'}$$

where $\mathbf{P}\mathbf{P}' = \mathbf{P}'\mathbf{P} = \mathbf{I}$ and $\underset{(k\times k)}{\mathbf{\Lambda}} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_k \end{bmatrix}$ with $\lambda_i > 0$

- inverse

$$\mathbf{A}^{-1} = \mathbf{P}\boldsymbol{\Lambda}^{-1}\mathbf{P}' = \sum_{i=1}^{k} \frac{1}{\lambda_i}\mathbf{e}_i\mathbf{e}_i'$$, $AA^{-1} = P\Lambda \underbrace{P'P}_{=I}\Lambda^{-1}P' = P\Lambda\Lambda^{-1}P' = PP' = I.$

- square-root matrix

$$\mathbf{A}^{1/2} = \sum_{i=1}^{k} \sqrt{\lambda_i}\,\mathbf{e}_i\mathbf{e}_i' = \mathbf{P}\boldsymbol{\Lambda}^{1/2}\mathbf{P}'$$

  - some properties

1. $(\mathbf{A}^{1/2})' = \mathbf{A}^{1/2}$ (that is, $\mathbf{A}^{1/2}$ is symmetric). $(A^{1/2})' = (P\Lambda^{1/2}P')' = P\overbrace{(\Lambda^{1/2})'}^{=\Lambda^{1/2}}P'$
2. $\mathbf{A}^{1/2}\mathbf{A}^{1/2} = \mathbf{A}$. $(P\Lambda^{1/2}P')(P\Lambda^{1/2}P') = P\Lambda^{1/2}\underbrace{P'P}_{=I}\Lambda^{1/2}P' = P\underbrace{\Lambda^{1/2}\Lambda^{1/2}}_{=\Lambda}P'$
3. $(\mathbf{A}^{1/2})^{-1} = \sum_{i=1}^{k} \frac{1}{\sqrt{\lambda_i}}\mathbf{e}_i\mathbf{e}_i' = \mathbf{P}\boldsymbol{\Lambda}^{-1/2}\mathbf{P}'$, where $\boldsymbol{\Lambda}^{-1/2}$ is a diagonal matrix with $1/\sqrt{\lambda_i}$ as the ith diagonal element.
4. $\mathbf{A}^{1/2}\mathbf{A}^{-1/2} = \mathbf{A}^{-1/2}\mathbf{A}^{1/2} = \mathbf{I}$, and $\mathbf{A}^{-1/2}\mathbf{A}^{-1/2} = \mathbf{A}^{-1}$, where $\mathbf{A}^{-1/2} = (\mathbf{A}^{1/2})^{-1}$.

- matrix inequalities and maximization

- **Cauchy–Schwarz Inequality.** Let $\mathbf{b}$ and $\mathbf{d}$ be *any* two $p \times 1$ vectors. Then

b d $b'd = \|b\|\|d\|\cos\theta$

$$(\mathbf{b}'\mathbf{d})^2 \le (\mathbf{b}'\mathbf{b})(\mathbf{d}'\mathbf{d})$$

with equality if and only if $\mathbf{b} = c\,\mathbf{d}$ (or $\mathbf{d} = c\,\mathbf{b}$) for some constant c.

Extended Cauchy–Schwarz Inequality. Let $\underset{(p\times 1)}{\mathbf{b}}$ and $\underset{(p\times 1)}{\mathbf{d}}$ be any two vectors, and let $\underset{(p\times p)}{\mathbf{B}}$ be a positive definite matrix. Then

$b'b$ $d'd$

$$(\mathbf{b}'\mathbf{d})^2 \le (\mathbf{b}'\mathbf{B}\,\mathbf{b})(\mathbf{d}'\mathbf{B}^{-1}\mathbf{d})$$

with equality if and only if $\mathbf{b} = c\,\mathbf{B}^{-1}\mathbf{d}$ (or $\mathbf{d} = c\,\mathbf{B}\,\mathbf{b}$) for some constant c.

- **Maximization Lemma.** Let $\underset{(p\times p)}{\mathbf{B}}$ be positive definite and $\underset{(p\times 1)}{\mathbf{d}}$ be a given vector. Then, for an arbitrary nonzero vector $\underset{(p\times 1)}{\mathbf{x}}$,

$$\max_{\mathbf{x}\neq\mathbf{0}} \frac{(\mathbf{x}'\mathbf{d})^2}{\mathbf{x}'\mathbf{B}\mathbf{x}} = \mathbf{d}'\mathbf{B}^{-1}\mathbf{d}$$

with the maximum attained when $\underset{(p\times 1)}{\mathbf{x}} = c\underset{(p\times p)(p\times 1)}{\mathbf{B}^{-1}\ \mathbf{d}}$ for any constant $c \neq 0$.

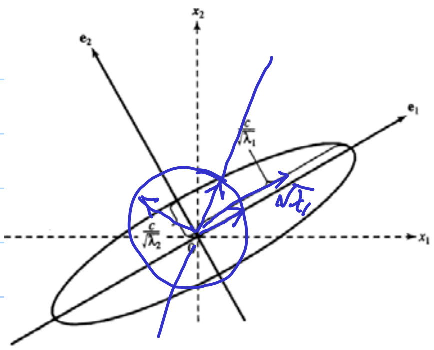
- **Maximization of Quadratic Forms for Points on the Unit Sphere.** Let $\underset{(p\times p)}{\mathbf{B}}$ be a positive definite matrix with eigenvalues $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_p \ge 0$ and associated normalized eigenvectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_p$. Then

e_1
$Be_1 = \lambda_1 e_1$
$e_1'Be_1 = \lambda_1 e_1'e_1$

$$\max_{\mathbf{x}\neq\mathbf{0}} \frac{\mathbf{x}'\mathbf{B}\mathbf{x}}{\mathbf{x}'\mathbf{x}} = \lambda_1 \qquad (\text{attained when } \mathbf{x} = \mathbf{e}_1)$$

$$\min_{\mathbf{x}\neq\mathbf{0}} \frac{\mathbf{x}'\mathbf{B}\mathbf{x}}{\mathbf{x}'\mathbf{x}} = \lambda_p \qquad (\text{attained when } \mathbf{x} = \mathbf{e}_p)$$

Let $y = cx$, $\frac{y'By}{y'y} = \frac{c^2x'Bx}{c^2x'x}$



Moreover,

$$\max_{\mathbf{x}\perp \mathbf{e}_1,\ldots,\mathbf{e}_k} \frac{\mathbf{x}'\mathbf{B}\mathbf{x}}{\mathbf{x}'\mathbf{x}} = \lambda_{k+1} \qquad (\text{attained when } \mathbf{x} = \mathbf{e}_{k+1}, k = 1, 2, \ldots, p-1)$$

where the symbol $\perp$ is read "is perpendicular to."

- approximation of a rectangular matrix by a lower-dimensional matrix
  - sum of squared differences

$$\sum_{i=1}^{m}\sum_{j=1}^{k}(a_{ij}-b_{ij})^2 = \text{tr}[(\mathbf{A}-\mathbf{B})(\mathbf{A}-\mathbf{B})'] \Leftarrow \text{generalization of } \cancel{\text{length}}\ \text{distance of vectors to matrices}$$

  - **Result 2A.16.** Let $\mathbf{A}$ be an $m \times k$ matrix of real numbers with $m \geq k$ and singular value decomposition $\underbrace{\mathbf{U\Lambda V'}}_{=\sum_{i=1}^{k}\lambda_i u_i v_i'}$. Let $s < k = \text{rank}(\mathbf{A})$. Then

$A: \mathbb{R}^k \to \mathbb{R}^m$, $m \geq k$

$$\mathbf{B} = \sum_{i=1}^{s} \lambda_i \boxed{\mathbf{u}_i \mathbf{v}_i'} \qquad \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0$$

dim(space spanned by columns of B) = S

is the rank-s least squares approximation to $\mathbf{A}$. It minimizes

$$\text{tr}[(\mathbf{A}-\mathbf{B})(\mathbf{A}-\mathbf{B})']$$

over all $m \times k$ matrices $\mathbf{B}$ having rank no greater than s. The minimum value, or error of approximation, is $\sum_{i=s+1}^{k} \lambda_i^2$.

- sample mean, covariance, and correlation as matrix operation
  - sample mean

$$\bar{\mathbf{x}} = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \vdots \\ \bar{x}_p \end{bmatrix} = \begin{bmatrix} \frac{\mathbf{y}_1'\mathbf{1}}{n} \\ \frac{\mathbf{y}_2'\mathbf{1}}{n} \\ \vdots \\ \frac{\mathbf{y}_p'\mathbf{1}}{n} \end{bmatrix} = \frac{1}{n}\begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{p1} & x_{p2} & \cdots & x_{pn} \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = \frac{1}{n}\mathbf{X}'\mathbf{1}$$

  - sample covariance matrix

$$\mathbf{1}\,\bar{\mathbf{x}}' = \frac{1}{n}\mathbf{1}\mathbf{1}'\mathbf{X} = \begin{bmatrix} \bar{x}_1 & \bar{x}_2 & \cdots & \bar{x}_p \\ \bar{x}_1 & \bar{x}_2 & \cdots & \bar{x}_p \\ \vdots & \vdots & \ddots & \vdots \\ \bar{x}_1 & \bar{x}_2 & \cdots & \bar{x}_p \end{bmatrix}$$

$$\underbrace{(I-\tfrac{1}{n}\mathbf{1}\mathbf{1}')X}\quad \mathbf{X} - \frac{1}{n}\mathbf{1}\mathbf{1}'\mathbf{X} = \begin{bmatrix} x_{11}-\bar{x}_1 & x_{12}-\bar{x}_2 & \cdots & x_{1p}-\bar{x}_p \\ x_{21}-\bar{x}_1 & x_{22}-\bar{x}_2 & \cdots & x_{2p}-\bar{x}_p \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1}-\bar{x}_1 & x_{n2}-\bar{x}_2 & \cdots & x_{np}-\bar{x}_p \end{bmatrix}$$

$$n\,\mathbf{S}_n = \left(\mathbf{X} - \frac{1}{n}\mathbf{1}\mathbf{1}'\mathbf{X}\right)'\left(\mathbf{X} - \frac{1}{n}\mathbf{1}\mathbf{1}'\mathbf{X}\right) = \mathbf{X}'\left(\mathbf{I} - \frac{1}{n}\mathbf{1}\mathbf{1}'\right)\mathbf{X} \Rightarrow S_n = \frac{1}{n}X'(I-\tfrac{1}{n}\mathbf{1}\mathbf{1}')X$$

since $\left(\mathbf{I} - \frac{1}{n}\mathbf{1}\mathbf{1}'\right)'\left(\mathbf{I} - \frac{1}{n}\mathbf{1}\mathbf{1}'\right) = \mathbf{I} - \frac{1}{n}\mathbf{1}\mathbf{1}' - \frac{1}{n}\mathbf{1}\mathbf{1}' + \frac{1}{n^2}\mathbf{1}\mathbf{1}'\mathbf{1}\mathbf{1}' = \mathbf{I} - \frac{1}{n}\mathbf{1}\mathbf{1}'$

- sample correlation matrix

$$\underset{(p\times p)}{\mathbf{D}^{1/2}} = \begin{bmatrix} \sqrt{s_{11}} & 0 & \cdots & 0 \\ 0 & \sqrt{s_{22}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sqrt{s_{pp}} \end{bmatrix} \qquad \underset{(p\times p)}{\mathbf{D}^{-1/2}} = \begin{bmatrix} \frac{1}{\sqrt{s_{11}}} & 0 & \cdots & 0 \\ 0 & \frac{1}{\sqrt{s_{22}}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{\sqrt{s_{pp}}} \end{bmatrix}$$

Since

s_{ii} s_{ij}

$$\mathbf{S}_n = \begin{bmatrix} s_{11} & s_{12} & \cdots & s_{1p} \\ \vdots & \vdots & \ddots & \vdots \\ s_{1p} & s_{2p} & \cdots & s_{pp} \end{bmatrix}$$

s_{jj}

$$\mathbf{R} = \begin{bmatrix} \frac{s_{11}}{\sqrt{s_{11}}\sqrt{s_{11}}} & \frac{s_{12}}{\sqrt{s_{11}}\sqrt{s_{22}}} & \cdots & \frac{s_{1p}}{\sqrt{s_{11}}\sqrt{s_{pp}}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{s_{1p}}{\sqrt{s_{11}}\sqrt{s_{pp}}} & \frac{s_{2p}}{\sqrt{s_{22}}\sqrt{s_{pp}}} & \cdots & \frac{s_{pp}}{\sqrt{s_{pp}}\sqrt{s_{pp}}} \end{bmatrix} = \mathbf{D}^{-1/2}\,\mathbf{S}_n\mathbf{D}^{-1/2}$$

❖ **Reading**: Textbook, 2.2, 2.3, 2.4, 2.7, Supplement 2A, 3.5