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- From the log-likelihood, we have $\hat{\mu}_{ij,MLE}$
  - for any $1 \leqslant i \leqslant I, 1 \leqslant j \leqslant J$, $\dfrac{\partial l}{\partial \mu_{ij}} = \dfrac{1}{\sigma^2}\left[\left(\sum_{k=1}^K Y_{ijk}\right) - K \times \mu_{ij}\right] = 0$
  - $\dfrac{\partial l}{\partial \sigma^2} = -\dfrac{\overset{N}{\boxed{IJK}}}{2\sigma^2} + \dfrac{1}{2\sigma^4}\sum_{i=1}^I\sum_{j=1}^J\sum_{k=1}^K (Y_{ijk} - \mu_{ij})^2 = 0 \rightarrow \hat{\sigma}^2_{MLE}$

Theorem 14 (UMVUE and MLE of the parameters in the model (□) in LNp.45)

$E(R_{ij}) = \sum_k \mu_{ij} = K\mu_{ij}$

- Let $R_{ij} = \sum_k Y_{ijk}$ for $1 \le i \le I, 1 \le j \le J$, and $R_\sigma = \sum_i\sum_j\sum_k Y_{ijk}^2$. Then,
$$(R_{11}, \ldots, R_{IJ}, R_\sigma)$$
is sufficient and complete (**Hint**. $(IJ+1)$-parameter exponential family) LNp.46
- $\overline{Y}_{ij\cdot} = R_{ij}/K$ is the UMVUE (by Lehmann-Scheffe Thm) and MLE of μ_{ij}
  - Note that $\hat{\mu}_{ij} \equiv \overline{Y}_{ij\cdot}$ is the *sample mean* of the (i,j)th sample $Y_{ij1}, \ldots, Y_{ijK}$
- By the *invariance property* of MLE, we have check LN,CH11,p.10

$\hat{\theta}_{MLE} \xrightarrow{e} \theta \curvearrowright g(\theta)$
$\Rightarrow$ MLE of $g(\theta)$ is $g(\hat{\theta}_{MLE})$

check LNp.40

- The MLE of $\overline{\mu}$ is
$$\hat{\overline{\mu}} = \frac{1}{IJ}\left(\sum_i\sum_j \hat{\mu}_{ij}\right) = \frac{1}{IJK}\left(\sum_i\sum_j\sum_k Y_{ijk}\right) \equiv \overline{Y}_{\cdots}$$
check tables in LNp.42 & 39
- The MLE of $\overline{\mu}_{i\cdot}$ is $\hat{\overline{\mu}}_{i\cdot} = \frac{1}{J}\left(\sum_j \hat{\mu}_{ij}\right) = \frac{1}{JK}\left(\sum_j\sum_k Y_{ijk}\right) \equiv \overline{Y}_{i\cdot\cdot}$, $i = 1, \ldots, I$

  The MLE of α_i is $\hat{\alpha}_i = \hat{\overline{\mu}}_{i\cdot} - \hat{\overline{\mu}} = \overline{Y}_{i\cdot\cdot} - \overline{Y}_{\cdots}$, $i = 1, \ldots, I$ check table in LNp.42

check LNp.41

- The MLE of $\overline{\mu}_{\cdot j}$ is $\hat{\overline{\mu}}_{\cdot j} = \frac{1}{I}\left(\sum_i \hat{\mu}_{ij}\right) = \frac{1}{IK}\left(\sum_i\sum_k Y_{ijk}\right) \equiv \overline{Y}_{\cdot j\cdot}$, $j = 1, \ldots, J$

  The MLE of β_j is $\hat{\beta}_j = \hat{\overline{\mu}}_{\cdot j} - \hat{\overline{\mu}} = \overline{Y}_{\cdot j\cdot} - \overline{Y}_{\cdots}$, $j = 1, \ldots, J$ check tables in LNp.42 & 39
- The MLE of δ_{ij} is
$$\hat{\delta}_{ij} = \hat{\mu}_{ij} - \hat{\overline{\mu}} - \hat{\alpha}_i - \hat{\beta}_j = \overline{Y}_{ij\cdot} - \overline{Y}_{i\cdot\cdot} - \overline{Y}_{\cdot j\cdot} + \overline{Y}_{\cdots}$$
for $i = 1, \ldots, I, j = 1, \ldots, J$ They are functions of R_{ij}'s ($\hat{\mu}_{ij}$'s)

Note. By Lehmann-Scheffe Thm, all these estimators are UMVUEs. (exercise) unbiased

- Connection between these estimators cf. table in LNp.39 & 42

$\delta_{11} \xleftarrow{e}$ $\hat{\delta}_{11} = \overline{Y}_{11\cdot} - (\hat{\overline{\mu}} + \hat{\alpha}_1 + \hat{\beta}_1)$ predition under MEM

$= \hat{\overline{\mu}} + \hat{\alpha}_1 + \hat{\beta}_1 + \hat{\delta}_{11}$

$\overline{Y}_{11\cdot} \xrightarrow{e} \mu_{11}$

A \ B	1	2	⋯	J	
1	$\overline{Y}_{11\cdot}$	$\overline{Y}_{12\cdot}$	⋯	$\overline{Y}_{1J\cdot}$	$\overline{Y}_{1\cdot\cdot} = \overline{Y}_{\cdots} + \hat{\alpha}_1$
⋮	⋮	⋮	⋮	⋮	⋮
I	$\overline{Y}_{I1\cdot}$	$\overline{Y}_{I2\cdot}$	⋯	$\overline{Y}_{IJ\cdot}$	$\overline{Y}_{I\cdot\cdot} = \overline{Y}_{\cdots} + \hat{\alpha}_I$
	$\overline{Y}_{\cdot 1\cdot} = \overline{Y}_{\cdots} + \hat{\beta}_1$	$\overline{Y}_{\cdot 2\cdot} = \overline{Y}_{\cdots} + \hat{\beta}_2$	⋯	$\overline{Y}_{\cdot J\cdot} = \overline{Y}_{\cdots} + \hat{\beta}_J$	$\overline{Y}_{\cdots}$

average average average average average

$\overline{Y}_{1\cdot\cdot} \xrightarrow{e} \overline{\mu}_{1\cdot}$, $\hat{\alpha}_1 \xrightarrow{e} \alpha_1$, $\overline{\mu}_{\cdot 1} \xleftarrow{e} \overline{Y}_{\cdot 1\cdot}$, $\beta_1 \xleftarrow{e} \hat{\beta}_1$, $\overline{Y}_{\cdots} \xrightarrow{e} \overline{\mu}$

unbiased estimator of σ^2

- Let $s_{ij}^2 = \frac{1}{K-1}\sum_k (Y_{ijk} - \overline{Y}_{ij\cdot})^2$ be the *sample variance* of the (i,j)th sample
  - (by Lehmann-Scheffe Thm) The *pooled* sample variance

Note 3 in LNp.11 & Def 1 in LN, CH11, p.7

$$s_p^2 = \frac{\sum_i \sum_j (K-1) s_{ij}^2}{IJ(K-1)} = \frac{1}{N-IJ}\left[\sum_i \sum_j \sum_k (Y_{ijk} - \overline{Y}_{ij\cdot})^2\right]$$

cf. $\xrightarrow{e} \sigma^2$ weights degree of freedom $\sum_i\sum_j\sum_k \hat{\varepsilon}_{ijk}^2$

is the UMVUE of σ^2, since (i) s_p^2 is unbiased, and (ii) $E(S_{ij}^2) = \sigma^2$ $\forall i, j$

$$(N-IJ)\, s_p^2 = \sum_i \sum_j \sum_k (Y_{ijk} - \overline{Y}_{ij\cdot})^2$$

$$= \left(\sum_i \sum_j \sum_k Y_{ijk}^2\right) - \left[\sum_i \sum_j (K\overline{Y}_{ij\cdot}^2)\right] = R_\sigma - \sum_i \sum_j (R_{ij}^2/K)$$

check LNp.47

The MLE of σ^2 is $\frac{N-IJ}{N} s_p^2 = \frac{1}{N}\left[\sum_i \sum_j \sum_k (Y_{ijk} - \overline{Y}_{ij\cdot})^2\right]$

mean 0 var σ^2

error ε_{ijk} cf. **Note.** $\hat{\epsilon}_{ijk} \equiv Y_{ijk} - \overline{Y}_{ij\cdot} = Y_{ijk} - \hat{\mu}_{ij}$ is called the *residual* of the (i,j,k)th observation.

$Y_{ijk} = \mu_{ij} + \varepsilon_{ijk}$ cf. cf. $= \hat{\mu}_{ij} + \hat{\varepsilon}_{ijk}$

Question 9. check Q8 (LNp.39) & Def 6 (LNp.44)

parameter space Ω $H_0^{(A)}$ $H_0^{(AB)}$ $H_0^{(B)}$

Under the model (□) in LNp.45, how to examine:

(i) whether the factor A has some effects on Y's, i.e.,

$$H_0^{(A)}: \alpha_1 = \cdots = \alpha_I = 0 \quad \text{vs.} \quad H_A^{(A)}: \text{at least one of } \alpha_i\text{'s is not } 0$$

(ii) whether the factor B has some effects on Y's, i.e.,

$$H_0^{(B)}: \beta_1 = \cdots = \beta_J = 0 \quad \text{vs.} \quad H_A^{(B)}: \text{at least one of } \beta_j\text{'s is not } 0$$

(iii) whether the factors A and B has some interactions on Y's, i.e.,

$$H_0^{(AB)}: \text{all } \delta_{ij}\text{'s are } 0 \quad \text{vs.} \quad H_A^{(AB)}: \text{at least one of } \delta_{ij}\text{'s is not } 0$$

Theorem 15 (Sum of squares decomposition) cf. SS decomposition for 1-way layout (LNp.7)

Consider the model (▽) in LNp.38. Define

no dist. assumption

If treat the IJ samples as a 1-way layout & perform 1-way ANOVA then, $SS_{within} = SS_E$, $SS_{between} = SS_A + SS_B + SS_{AB}$

total variation in the data Y_{ijk}'s

$$SS_{TOT} = \sum_i \sum_j \sum_k (Y_{ijk} - \overline{Y}_{\cdots})^2 = \sum_i \sum_j \sum_k (Y_{ijk} - \hat{\mu})^2$$

Then, the SS_{TOT} can be decomposed to

$$\underbrace{\sum_i \sum_j \sum_k (Y_{ijk} - \overline{Y}_{\cdots})^2}_{SS_{TOT}} = \underbrace{\sum_i \sum_j \sum_k (Y_{ijk} - \overline{Y}_{ij\cdot})^2}_{SS_E \left(= \sum_i \sum_j \sum_k \hat{\epsilon}_{ijk}^2 = (N-IJ)\, s_p^2\right)}$$

$s_p^2 \xrightarrow{e} \sigma^2$

$$+ \underbrace{JK\left[\sum_i (\overline{Y}_{i\cdot\cdot} - \overline{Y}_{\cdots})^2\right]}_{SS_A \left(= JK\left(\sum_i \hat{\alpha}_i^2\right)\right)} + \underbrace{IK\left[\sum_j (\overline{Y}_{\cdot j\cdot} - \overline{Y}_{\cdots})^2\right]}_{SS_B \left(= IK\left(\sum_j \hat{\beta}_j^2\right)\right)}$$

$\sum_i\sum_j\sum_k \hat{\alpha}_i^2$ $\sum_i\sum_j\sum_k \hat{\beta}_j^2$

check the table in LNp.48

$$+ \underbrace{K\left[\sum_i \sum_j (\overline{Y}_{ij\cdot} - \overline{Y}_{i\cdot\cdot} - \overline{Y}_{\cdot j\cdot} + \overline{Y}_{\cdots})^2\right]}_{SS_{AB} \left(= K\left(\sum_i \sum_j \hat{\delta}_{ij}^2\right)\right)}$$

$\sum_i\sum_j\sum_k \hat{\delta}_{ij}^2$

source of variation

That is,

variation of 隨機 (∵ ε_{ijk}'s are r.v.'s)

$$SS_{TOT} = SS_E + \underset{\mathrm{I}}{SS_A} + \underset{\mathrm{II}}{SS_B} + \underset{\mathrm{III}}{SS_{AB}}$$

variation of 規律 (∵ α_i's, β_j's, δ_{ij}'s: parameters) — A, B, AB

Proof. This proof is similar to that of one-way ANOVA (LNp.7). This SS identity can be proved by writing ← This proof requires no dist. assumption